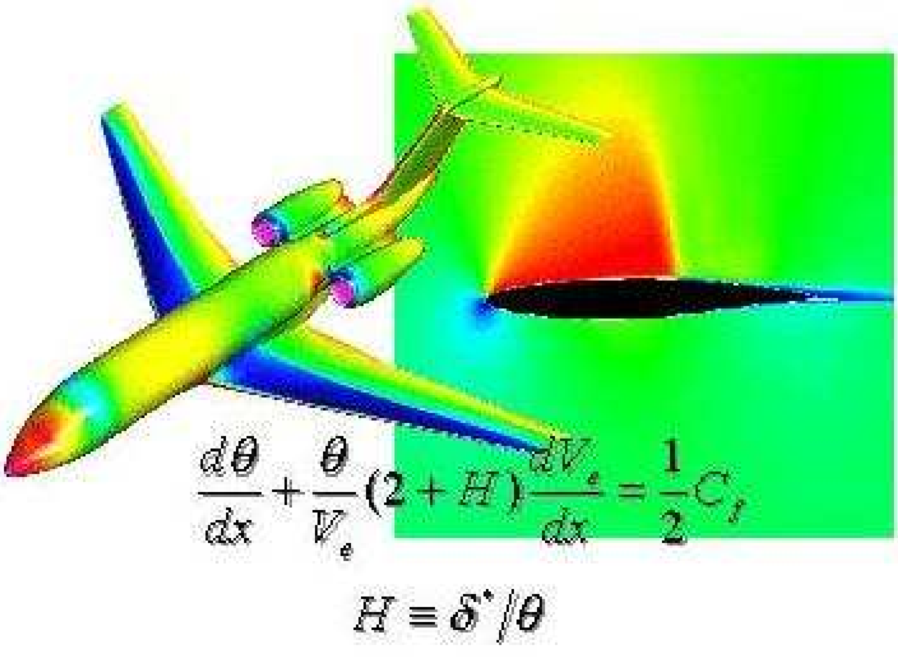
Supersonic Thin Airfoil Theory



AA200b

Lecture 5

January 20, 2005

Foundations of Supersonic Thin Airfoil Theory

The equations that govern the steady subsonic and supersonic flow of an inviscid fluid around *thin* bodies can be shown to be very similar in appearance to Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{1}$$

which we have used in previous lectures for incompressible flow. In fact, the equations for a free stream Mach number different from zero are given by

$$(1 - M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{2}$$

Notice that although Eqns. 1 and 2 look very similar, a fundamental change in behavior occurs as the free stream Mach Number, M_∞, increases past the value of 1: the PDE goes from *elliptic* to *hyperbolic* and it takes on a *wavelike* behavior.

Eqn. 2 governs the behavior of a supersonic flow where only *small* disturbances have been introduced.

You should already be familiar with a number of results for two-dimensional, isentropic, supersonic flow such as the Prandtl-Meyer function (weak reversible turning analysis, both positive and negative). These notes expand on that knowledge an apply it to the computation of forces and moments on supersonic airfoils.

The main difference when compared to incompressible thin airfoil theory is the appearance of drag (in an inviscid flow). The source of this drag is the *radiation* to infinity of pressure waves created by disturbances at the surface of the airfoil. We will call this drag *wave drag*.

Use of 3D Weak-Turning Wave Theory

It can be shown that for small angle changes, either compressive or expansive, one can relate flow speed and static pressure changes to the infinitesimal flow angle changes:

$$\begin{aligned} \frac{du}{u} &= -\frac{d\theta}{\sqrt{M^2-1}} \\ \frac{dp}{p} &= +\frac{\gamma M^2 d\theta}{\sqrt{M^2-1}}. \end{aligned} \tag{3}$$

The idea of thin airfoil theory is to use these expressions directly for *finite-but-small* angle changes imposed on the incoming flow by airfoil surface shapes and angle of attack. Thus, if the airfoil is thin enough and the angle of attack is small enough, one may expect to be able to *approximate* flow property variations in the forms

$$\begin{aligned} \frac{\Delta w}{w} &= -\frac{\Delta\theta}{\sqrt{M_\infty^2 - 1}} \\ \frac{\Delta p}{p} &= +\frac{\gamma M_\infty^2 \Delta\theta}{\sqrt{M_\infty^2 - 1}}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Delta p &\equiv p - p_\infty \\ \Delta w &\equiv w - U_\infty, \end{aligned}$$

and where, if (u, v) are the (x, y)-components of the 2D velocity then $w \equiv \sqrt{u^2 + v^2}$. Also, in these expressions, the local Mach number, M, has been replaced by M_∞, the incoming Mach number, since the variations in M are themselves expected to be *small* for thin airfoils at low angles of attack. The situation is illustrated in Fig. 1 for a thin biconvex airfoil at zero angle of attack. Addition of a small angle of attack α of order $\tan^{-1}\frac{t}{c} \sim \frac{t}{c}$ would not change the figure substantially.

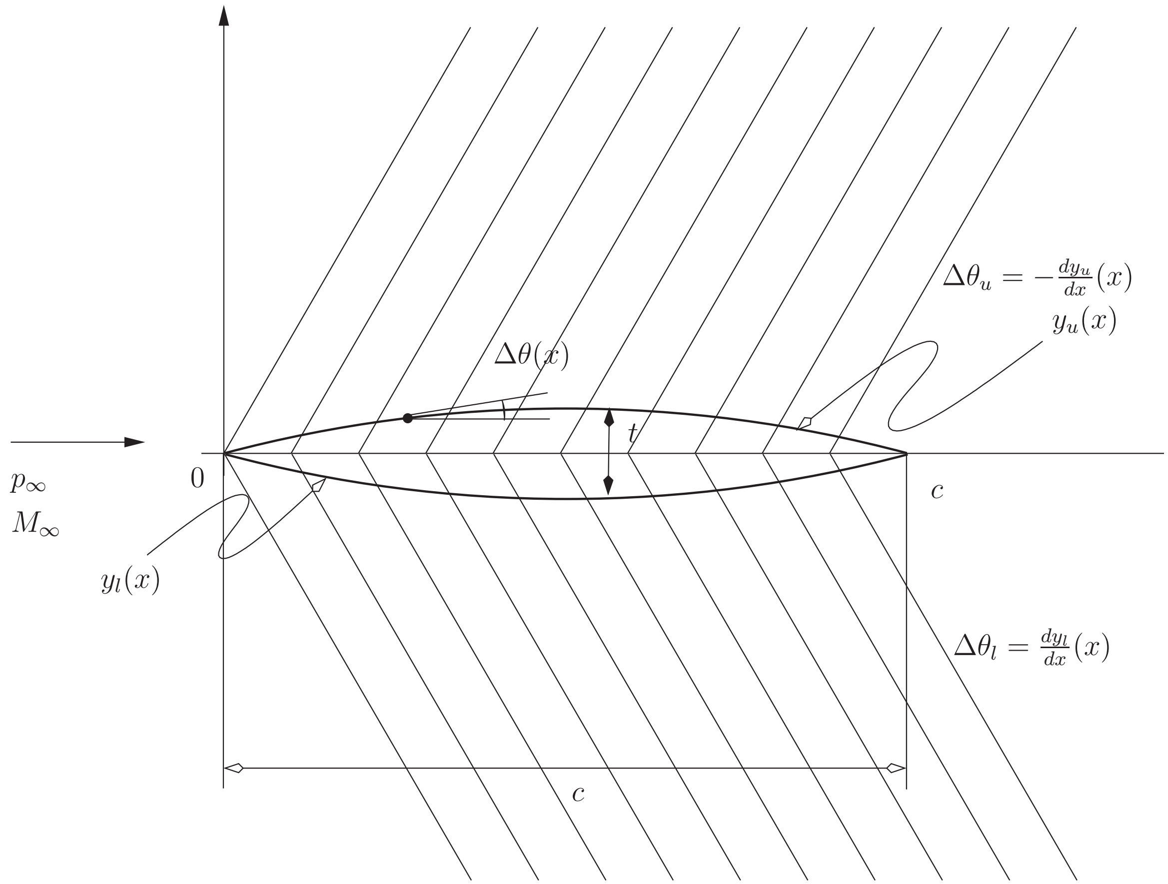


Figure 1: Thin Curved Airfoil in Supersonic Flow.

In order for the airfoil to be *thin* we require that $\tan^{-1}\frac{t}{c} \sim \frac{t}{c}$, i.e.

$$\frac{t}{c} << 1,$$

where t and c are illustrated in the figure, and, of course, if there were an angle of attack we would also require that

$$\alpha << 1 \text{ (radians).}$$

The condition that $\Delta\theta$ is everywhere small implies further that, in thin airfoil applications

$$\tan\Delta\theta = \frac{v}{u} \sim \frac{v}{U_\infty} << 1, \tag{5}$$

whereas Eqn. 4, together with Eqn. 5, implies that $\frac{\Delta w}{w} \sim \frac{\Delta u}{w} \sim \frac{\Delta u}{U_\infty} << 1$. Thus, in thin airfoil applications one need not distinguish between u and w

as they occur in Eqns. 4-5. It is also true in the same limit that

$$\begin{aligned} \frac{\Delta\rho}{\rho} &= \frac{M_\infty^2 \Delta\theta}{\sqrt{M_\infty^2 - 1}} \\ \frac{\Delta c}{c} &= \frac{\gamma - 1}{2} M_\infty^2 \frac{\Delta\theta}{\sqrt{M_\infty^2 - 1}}, \end{aligned} \tag{6}$$

and

$$\frac{\Delta M}{M} = -\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) \frac{\Delta\theta}{\sqrt{M_\infty^2 - 1}}. \tag{7}$$

In all of these expressions, $\Delta\theta$ is to be taken positive in the compressive sense. Also, because $\Delta\theta$ must be small for thin airfoil theory to hold, we can write

$$\sin\Delta\theta \approx \tan\Delta\theta \approx \Delta\theta, \quad \cos\Delta\theta \approx 1$$

when convenient. Throughout this approximate theory any given quantity is known or can be calculated only to its lowest order in $\Delta\theta$. A variation in pressure, for example, is linear in $\Delta\theta$; terms of order $(\Delta\theta)^2$ and higher can and should be consistently neglected. Similarly, a quantity such as airfoil drag is inherently a product of two linear quantities and thus of order $(\Delta\theta)^2$; for such quantities, terms of order $(\Delta\theta)^3$ and higher are to be neglected.

Airfoil Performance: Approximate Thin-Airfoil Version

As sketched in Fig. 1, the upper and lower surfaces of the airfoil are located at

$$y = y_u(x)$$
$$y = y_l(x).$$

On the upper surface, $\Delta\theta$ is positive (compressive) for $y_u(x)$ increasing; thus, on that surface

$$\Delta\theta_u = \frac{dy_u}{dx}.$$

On the lower surface, by contrast, $\Delta\theta$ is in the compressive sense when $y_l(x)$ becomes increasingly negative. In that case

$$\Delta\theta_l = -\frac{dy_l}{dx}.$$

Using Eqn. 5 together with the expressions above, we have that

$$\frac{\Delta p_u}{p_\infty} = \frac{p_u - p_\infty}{p_\infty} = \frac{\gamma M_\infty^2}{\sqrt{M_\infty^2 - 1}} \frac{dy_u}{dx} \quad \text{(upper surface)} \tag{8}$$

$$\frac{\Delta p_l}{p_\infty} = \frac{p_l - p_\infty}{p_\infty} = \frac{\gamma M_\infty^2}{\sqrt{M_\infty^2 - 1}} \frac{dy_l}{dx} \quad \text{(lower surface)}$$

In aerodynamic applications it is customary to introduce the dimensionless pressure coefficient, C_p, defined by

$$C_p \equiv \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}. \tag{9}$$

Note that

$$\frac{1}{2}\rho_\infty U_\infty^2 = \frac{\gamma M_\infty^2}{2} p_\infty,$$

and therefore, for $M_\infty \neq 0$, one can relate $\frac{\Delta p}{p_\infty}$ and the pressure coefficient conveniently by

$$C_p = \frac{2}{\gamma M_\infty^2}\frac{\Delta p}{p_\infty}.$$

Applied to the pressure variations on the upper and lower surfaces of the airfoil, this gives

$$\begin{aligned} C_{p_u} &= \frac{2}{\sqrt{M_\infty^2 - 1}}\frac{dy_u}{dx} \qquad (10) \\ C_{p_l} &= -\frac{2}{\sqrt{M_\infty^2 - 1}}\frac{dy_l}{dx}. \end{aligned}$$

Knowing the approximate surface pressures, this puts us in a position to be able to calculate both the lift and the drag of the airfoil, to the same approximation. For example, an element at x on the upper surface, of area

$\frac{dx}{\cos(\Delta\theta_u(x))}$ per unit depth into the page, contributes an amount

$$dl_u = -p_u \left(\frac{dx}{\cos \Delta\theta_u} \right) \cos \Delta\theta_u = -p_u dx \tag{11}$$

to the airfoil lift per unit span. Similarly, an element on the lower surface contributes

$$dl_l = +p_l dx.$$

Thus, the lift/unit span is

$$l = \int_0^c \left(p_l(x) - p_u(x) \right) dx. \tag{12}$$

In coefficient form, this can be shown to be

$$C_l \equiv \frac{1}{c} \int_0^c \left(C_{p_l}(x) - C_{p_u}(x) \right) dx \tag{13}$$

Using our approximate formulas for C_{p_u} and C_{p_l} we can get an explicit expression for the thin-airfoil lift coefficient.

$$C_l = -\frac{2}{\sqrt{M_\infty^2 - 1}}\frac{1}{c}\int_0^c \left(\frac{dy_l}{dx} + \frac{dy_u}{dx}\right) dx$$
$$= -\frac{2}{\sqrt{M_\infty^2 - 1}}\left[\frac{y_l(c) - y_l(0)}{c} + \frac{y_u(c) - y_u(0)}{c}\right],$$

or, since $y_l(c) = y_u(c)$ and $y_l(0) = y_u(0)$ for a closed-contour airfoil

$$C_l = -\frac{4}{\sqrt{M_\infty^2 - 1}}\frac{y_u(c) - y_u(0)}{c} = +\frac{4}{\sqrt{M_\infty^2 - 1}}\alpha, \qquad (14)$$

since $\frac{y_u(c)-y_u(0)}{c} = \tan\alpha \approx \alpha$. The angle of attack α is in radians. This is a classical result in aerodynamics and provides a measure, for supersonic airfoils, of achievable lift for wings at small angles of attack. This approximation works rather well, even when extended to three dimensions, to predict the lift curve slope of a wing, particularly one where the sections behave in a two-dimensional fashion (issues of subsonic and supersonic leading edges and regions of influence need to be considered here).

A review of the derivation clearly shows that the lift coefficient is independent of the airfoil profile shape: the lift coefficient in supersonic flow, for a given angle of attack α is the same for a flat plate, a diamond airfoil, or a biconvex airfoil.

Wave Drag Calculation

We have just seen that in supersonic thin airfoil theory, the the lift coefficient is *independent* of airfoil shape. Airfoil drag, however, is another matter; this depends strongly on the shape of the airfoil. To obtain its value, note that an element on the upper surface contributes to the drag amount (per unit span)

$$d(\text{drag}) = p_u \frac{dx}{\cos \Delta\theta_u} \sin \Delta\theta_u = p_u \tan \Delta\theta_u dx = p_u \frac{dy_u}{dx} dx \quad \text{(upper surface)},$$

and, similarly

$$d(\text{drag}) = -p_l \frac{dy_l}{dx} dx \quad \text{(lower surface)}.$$

Integrating over the chord we obtain

$$\text{drag/unit span} = \int_0^c \left(p_u \frac{dy_u}{dx} - p_l \frac{dy_l}{dx} \right) dx,$$

which in coefficient form can be shown to be

$$C_d = \frac{1}{c} \int_0^c \left(C_{p_u} \frac{dy_u}{dx} - C_{p_l} \frac{dy_l}{dx} \right) dx. \tag{15}$$

Using our expressions from the thin airfoil limit we can show

$$C_d = \frac{2}{\sqrt{M_\infty^2 - 1}} \frac{1}{c} \int_0^c \left[\left(\frac{dy_u}{dx} \right)^2 + \left(\frac{dy_u}{dx} \right)^2 \right] dx \quad \text{(thin airfoil)} \tag{16}$$

For the case of a flat plate at an angle of attack α

$$C_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}, \tag{17}$$

while for a diamond airfoil with half angle τ

$$C_d = \frac{4(\alpha^2 + \tau^2)}{\sqrt{M_\infty^2 - 1}}. \tag{18}$$

For a curved biconvex airfoil, C_d must be calculated for each specified shape.

Notice that the C_d of a thin airfoil in supersonic flow can be shown to be proportional to the square of the thickness-to-chord ratio. This is primarily the reason why airfoils in supersonic aircraft are made so thin! In addition, one must consider the effect of the shape of the airfoil in the drag calculation carefully. This leads to very interesting optimizations in inviscid flow (SSTs, supersonic business jets, boom minimization problem, etc.)

Location of the Center of Pressure in Supersonic Flow

One of the main differences between subsonic and supersonic flow is that the center of pressure for a flat plate airfoil is no longer located at the quarter chord location. In fact, as the flow transitions from subsonic to supersonic, the center of pressure moves backwards to the $\frac{c}{2}$ location. This can be simply shown by the following definition of the aerodynamic center:

$$L^{'} x_{cp} + M^{'}_{le} = 0,$$

which can be used to solve for the location of the center of pressure as

$$x_{cp} = \frac{M^{'}_{le}}{L^{'}}.$$

By definition of these quantities, one can write:

$$\frac{x_{cp}}{c} = \frac{\frac{1}{c}\int_0^c x(p_l - p_u)dx}{\int_0^c (p_l - p_u)dx} = \frac{\frac{1}{c}\int_0^c x(C_{p_l} - C_{p_u})dx}{\int_0^c (C_{p_l} - C_{p_u})dx},$$

and using the results for C_{p_u} and C_{p_l} from thin airfoil theory

$$\frac{x_{cp}}{c} = \frac{\frac{1}{c}\int_0^c x(\frac{dy_l}{dx} + \frac{dy_u}{dx})dx}{\int_0^c (\frac{dy_l}{dx} + \frac{dy_u}{dx})dx}.$$

For a flat plate at an angle of attack α, one can easily show that $\frac{dy_l}{dx} = \frac{dy_l}{dx} = -\tan\alpha \approx -\alpha$, and therefore

$$\frac{x_{cp}}{c} = \frac{\frac{1}{c}\int_0^c x dx}{\int_0^c dx} = \frac{1}{2},$$

which locates the center of pressure at the *mid-chord* location. All airplanes that have to fly supersonically must reach their supersonic speeds by going through the subsonic regime. The large shift in the location of the center of pressure (similar conclusions can be drawn for the aerodynamic center) poses significant challenges for the longitudinal stability of this kind of aircraft. In fact, the Concorde uses a fairly complex system to pump fuel forward/aft in order to maintain the proper location of the center of gravity so that a sufficient static margin is achieved.

Optimizing Lift-to-Drag Ratio

It is apparent from a comparison of the drag coefficients for a flat plate airfoil and a diamond airfoil at a given Mach Number and at fixed α that there is no optimum wedge angle (or maximum t/c ratio) than simply $\tau = 0$. Any τ at all increases the drag relative to the flat plate case. This is generally true even for more complicated airfoil shapes: at fixed M_∞ and angle of attack (i.e., fixed lift coefficient) any airfoil thickness increases the drag; the flat plate is the minimum-drag shape for a given lift. Of course, a flat plate is not a viable option in practical wing or aircraft design.

A more interesting question is the following: given an airfoil shape (for example, the diamond airfoil with specified τ), at what angle of attack, α, is the lift-to-drag ratio a maximum? We have in that case

$$\frac{L}{D} = \frac{C_l}{C_d} = \frac{\alpha}{\alpha^2 + \tau^2},$$

from our previous results. It is easy to show from this that the optimum α is at $\alpha = \tau$ and the corresponding maximum lift-to-drag ratio is

$$\left.\frac{L}{D}\right)_{max} = \frac{1}{2\tau},$$

with τ expressed in radians.