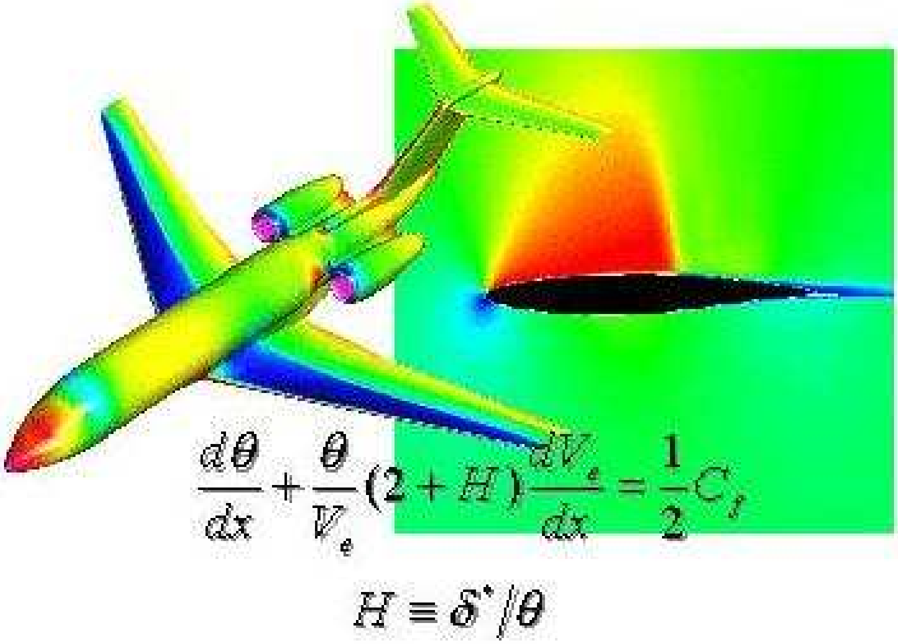
The Boundary Layer Approximation



AA200b

Lecture 6

January 21, 2005

Two-Dimensional Navier-Stokes Equations

The governing equations for the flow of a two-dimensional incompressible fluid can be summarized as follows

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) &= -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned} \tag{1}$$

with the additional incompressibility constraint being given by

$$\nabla\cdot\mathbf{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2}$$

Equations 1 are usually called the *Navier-Stokes Equations* and they completely govern the behaviour of a Newtonian fluid (subject to the appropriate boundary conditions and taking into account the fact that the solution may be unsteady). If we were able to solve these equations efficiently and robustly, there would be no need for simplifications such as the Euler equations or the potential flow equation.

Unfortunately, there are only a few analytic solutions to the N-S equations. Even for the simplest flows (geometrically speaking) analytic solutions are hard (or impossible) to obtain. The alternatives are to either solve these equations numerically, or to obtain simplifications to the N-S equations while retaining some of the *characteristics* of the original equations.

Fortunately, although the various terms in Equations 1 can be difficult to calculate, quite a few of them are usually very small for typical applied aerodynamics applications and can be neglected altogether. In order to identify the terms that can be neglected, we perform an *order of magnitude analysis* whose objective is to identify the *rough* magnitude of the various terms in the N-S equations and to drop those that are small compared to the other terms in the equations.

Note that these estimates are rough (coud be in error by factors of $5 - 10$) but, for large Reynolds numbers we will be able to justify their elimination, since the upper bound on their magnitude will still be much smaller than the values of the other terms in the equations.

No-Slip Boundary Condition

Our derivation is based on the observation that the proper boundary condition in a viscous flow at a solid wall is the *no-slip* condition. Empirically, it has been observed that a viscous flow *sticks* to the surface of the body. In a frame of reference where the body is stationary, this boundary condition reduces to

$$\mathbf{V} = 0. \tag{3}$$

Notice that in inviscid flows, the only boundary condition that we have specified is the *flow-tangency* boundary condition. In viscous flows, the order of the partial differential equation increases by one requiring us to impose a condition on the *tangential velocity component*, thus enforcing the no-slip condition.

Now, although the Reynolds number of the flow will give us an indication of the value of the ratio of inertial terms vs. viscous terms, and although this number is typically large for flows of aerodynamic interest, there must be some areas of the flow where the *effective Reynolds number* is much smaller than that which would be calculated using the reference length of the body of interest. Otherwise, the viscous terms would be negligible *everywhere*, the inviscid approximation would be uniformly valid, and it would be impossible to satisfy the *no-slip* boundary condition.

Boundary Layer - Order of Magnitude Analysis

Fundamental to the estimation of the values of the various terms in Equations 1 is the definition of a *characteristic length scale*, the distance that one needs to move in each of the coordinate directions in order to see a pre-specified change in the value of the variable of interest.

Empirically it is found that the characteristic lengths in the x and y coordinate directions differ substantially in magnitude. For the purposes of this discussion we will assume, as can be seen in Figure 1 below, that the wall is aligned with the x-axis. The y-axis is in the direction normal to the wall. The boundary layer will develop along the x coordinate.

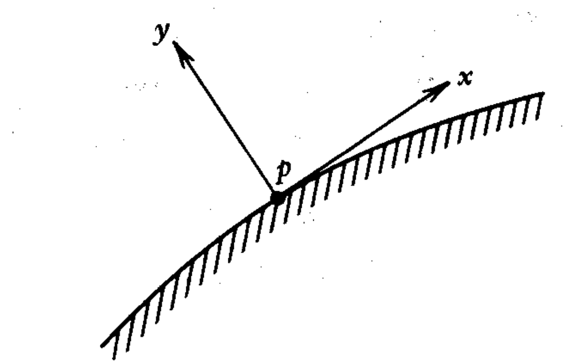


Figure 1: Coordinate System for Boundary Layer Analysis

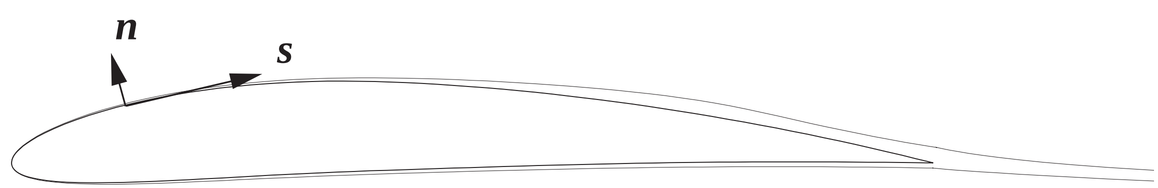


Figure 2: Typical Boundary Layer on a NACA 4410 airfoil

For a typical airfoil or wing shape, as we can see in Figure 2 we will locally define a coordinate that is tangential to the surface of the airfoil, s, and one that runs perpendicular to the surface of the airfoil, n, at every point along the upper and lower surfaces.

We will further assume that typical changes in the relevant variables occur with a characteristic length, $L \sim c$, in the x-direction, while they occur with characteristic length, δ, in the y-direction. The value of δ will be associated with the boundary layer thickness which will end up being a small quantity when compared with the convective length scale, L.

Now, for this small region of the flow called the *boundary layer* to exist, the characteristic lengths L and δ must be in such proportions that the viscous terms are just as important as the inertia terms inside of it. Outside the boundary layer, the viscous terms can be neglected all together and the inviscid approximation is perfectly valid.

Let V_e be the flow speed just outside of the boundary layer. Then

$u \approx V_e$ for $y \approx \delta$, and therefore,

$$u \sim V_e.$$

Since V_e vanishes at the stagnation points, the change in u in the x-direction can be as large as V_e, and

$$\frac{\partial u}{\partial x} \sim \frac{V_e}{L}.$$

Since, from the *no-slip* boundary condition, $u = 0$ at $y = 0$, the change of u in the y direction can also be as large as V_e and then,

$$\frac{\partial u}{\partial y} \sim \frac{V_e}{\delta}.$$

From the continuity equation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \sim \frac{V_e}{L}.$$

However, the order of magnitude analysis for this term is

$$\frac{\partial v}{\partial y} \sim \frac{\Delta v}{\delta},$$

and putting these two last equations together, together with the fact that $v = 0$ at $y = 0$, we can easily see that

$$v \sim \Delta v \sim \frac{\delta}{L} V_e$$

within the boundary layer. One can immediately see that the u component of velocity is much larger than the v component within the boundary layer.

We are now ready to estimate the other terms in the N-S equations as follows:

$$
\begin{aligned}
\rho u \frac{\partial u}{\partial x} &\sim \rho \frac{V_e^2}{L} \\
\rho v \frac{\partial u}{\partial y} &\sim \rho \frac{\delta}{L} V_e \frac{V_e}{\delta} = \rho \frac{V_e^2}{L} \\
\mu \frac{\partial^2 u}{\partial x^2} &\sim \mu \frac{V_e}{L^2} \\
\mu \frac{\partial^2 u}{\partial y^2} &\sim \mu \frac{V_e}{\delta^2} \\
\rho u \frac{\partial v}{\partial x} &\sim \rho V_e \frac{\delta}{L} \frac{V_e}{L} = \rho \frac{V_e^2 \delta}{L^2} \\
\rho v \frac{\partial v}{\partial y} &\sim \rho \frac{\delta}{L} V_e \frac{\delta}{L} \frac{V_e}{\delta} = \rho \frac{V_e^2 \delta}{L^2}
\end{aligned}
$$

$$\mu \frac{\partial^2 v}{\partial x^2} \sim \mu \frac{\delta}{L} \frac{V_e}{L^2}$$
$$\mu \frac{\partial^2 v}{\partial y^2} \sim \mu \frac{\delta}{L} \frac{V_e}{\delta^2} = \frac{\mu V_e}{L\delta}.$$

So far we have not been able to say anything about the terms including directional derivatives of the pressure, p, since, in order to estimate terms containing its derivatives, we must know something about *typical* changes in its value, Δp, which are not as easy to estimate as the changes in the velocity components, u, and v. However, we know that these pressure gradient terms must be balanced by the other terms in the equation, and we will use this knowledge to find estimates of their magnitude.

From the estimates above, we can conclude that

1. The two inertia terms are of the same order of magnitude within

the boundary layer, and therefore we must carry both terms in the simplification of the N-S equations.

2. The viscous term involving x derivatives is *much* smaller than the one involving y derivatives, assuming that $\delta \ll L$.

3. Within the boundary layer, the largest of the viscous terms is comparable in magnitude to the inertial terms if

$$\rho \frac{V_e^2}{L} \sim \frac{\mu}{\delta^2} V_e$$

or in other words

$$\frac{\delta}{L} \sim \sqrt{\frac{\mu}{\rho V_e L}} = \frac{1}{\sqrt{Re}}$$

is the estimate of the boundary layer thickness necessary to make viscous and inertia terms of the same magnitude within the boundary layer itself.

Now for the more challenging part of the estimate: the pressure gradient terms. Following the derivations in Schlichting and Kuethe and Chow, the pressure terms must be at most of the same order of magnitude as both the dominant viscous term and the inertia terms in both of Equations 1. This statement has different implications for the x- and y-momentum equations.

In the x-momentum equation, the conclusion is simply that

$$\frac{\partial p}{\partial x} \sim \rho \frac{V_e^2}{L},$$

which cannot be neglected with respect to the other terms in the equation and therefore must be retained. Notice that, on the basis of the estimate we are unable to extract additional information that may lead to neglecting this or other terms altogether: their value is of magnitude $\rho V_e^2 / L$ which is certainly not a small quantity.

The conclusions that can be derived for the y-momentum equation are quite different. Under the assumption stated above that

$$\frac{\delta}{L} = \frac{1}{\sqrt{Re}},$$

both the dominant viscous term, $\mu\frac{\partial^2 v}{\partial y^2}$, and the inertial terms are of the same magnitude. Since these two terms balance the pressure gradient term, the upper bound on the value of $\partial p/\partial y$ must be identical

$$\frac{\partial p}{\partial y} \sim \frac{\delta}{L}\rho\frac{V_e^2}{L}.$$

Otherwise, the pressure gradient terms could be unbalanced.

Therefore, as in the x-momentum equation, all three terms are of the same magnitude. Does this mean we should keep all terms as we did with the x-momentum equation? Certainly. However, the difference lies in the fact that the estimate of the magnitude of these terms has been shown to be $(\delta/L)\rho(V_e^2/L)$ which is *much* smaller than the magnitude of the equivalent terms in the x-momentum equation (which had magnitude $\rho V_e^2/L$) by a factor of δ/L. If this value of δ/L is *small* we may then make an engineering approximation for the whole y-momentum equation. We may consider *any* of the terms in the equation to be negligible. That is,

$$\rho u \frac{\partial v}{\partial x} \sim \rho v \frac{\partial v}{\partial y} \sim \frac{\partial p}{\partial y} \sim \mu \frac{\partial^2 v}{\partial y^2} \sim \frac{\delta}{L} \rho \frac{V_e^2}{L} \quad \text{are small.}$$

However, the most attractive term that we can set to zero is the pressure gradient one, since it provides immediate information regarding the fact that the static pressure remains constant across the boundary layer. Moreover, this observation will allow us to produce boundary layer-inviscid coupled methods that can be very useful in practice. Since the normal pressure gradient is found to be zero in the limit of very high Reynolds numbers, the pressure on the wall is effectively equal to the pressure at the edge of the boundary layer, which can be calculated using an inviscid method. Because of this situation, the pressure is said to be "impressed" on the boundary layer by the outer inviscid flow. Therefore, in the solution of the Boundary Layer equations, we will consider p, the static pressure to be a known quantity.

The N-S equations can then be simplified to

$$\begin{aligned} \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial p}{\partial y} &\approx 0, \end{aligned} \tag{4}$$

which, together with the continuity equation yield a subset called the

Boundary Layer Equations. Note that we could have just as well said that a valid reduction of the N-S equations, according to the order of magnitude analysis performed earlier, could be

$$\begin{aligned} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \\ \rho u\frac{\partial v}{\partial x} &\approx 0, \end{aligned} \tag{5}$$

although the usefulness of the second equation can be *easily* put in question (despite the fact that it is indeed true!!!)

The Boundary Layer equations are valid *inside* the boundary layer for large Reynolds numbers (so that δ/L is indeed a small quantity). In addition, the fact that the normal pressure gradient is negligible assumes that the surface of the wall has very low curvature. If this is not the case, the centripetal correction of Cebeci and Bradshaw can be introduced to give a slightly modified second equation

$$\frac{\partial p}{\partial y} = -\rho \frac{u^2}{R},$$

where u is the component of velocity tangential to the wall and R is the local surface radius of curvature.