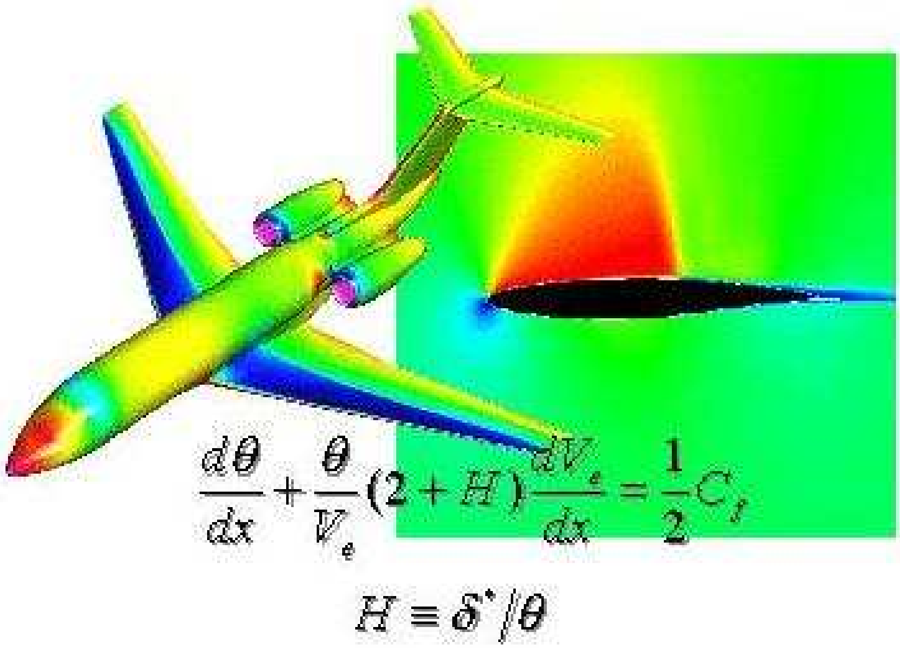
Solutions to the Boundary Layer Equations



AA200b

Lecture 7

January 25, 2005

Boundary Layer Equations

In the previous lecture we saw how using an *order of magnitude analysis* analysis we were able to simplify the two-dimensional Navier-Stokes equations to the following set of governing equations for an incompressible fluid

$$\begin{aligned} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial p}{\partial y} &\approx 0 \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{1}$$

where the first two equations are the simplified x- and y-momentum equations, while the last equation is the statement of continuity.

The boundary conditions to be imposed on these equations are simply the *no-slip* boundary condition at the solid surface situated a $y = 0$,

$$\mathbf{V} = 0 \quad \text{at} \quad y = 0$$

and the fact that at the edge of the boundary layer, the x-component of the velocity vector must be equal to V_e. Furthermore, note that the pressure gradient term $\frac{dp}{dx}$ is treated as a known quantity since it is dictated by the outer inviscid flow. In order to relate the value of this pressure gradient at the edge of the boundary layer, to the velocity there, $V_e(x)$, we can simply differentiate the Bernoulli equation with respect to the x-direction to obtain

$$\frac{dp}{dx} = -\rho V_e \frac{dV_e}{dx} \tag{2}$$

Finally, keep in mind that the boundary layer equations are only valid in the limit of $Re \rightarrow \infty$. Moreover, we had also implicitly assumed that $u \gg v$, which implies that in the neighborhood of separation points, where the tangential component of velocity, $u = 0$, the boundary layer assumption will also break down.

Solution Alternatives

The main reason why we sought a simplification to the N-S equations was that we hoped we could obtain a more easily solvable set of equations. We should now be concerned with the type of problems for which these equations can be solved. As usual, we have two main alternatives:

- Find analytic solutions to the boundary layer equations, and extract valuable information that we may use to comprehend the nature of the flow.

- Use numerical methods to obtain solutions that will usually be more practical, but that will hide *analytic* details about the solution.

In AA200b, we will assume that you have had some exposure to the first approach, but that are not very familiar with the second.

Exact Solutions to the Boundary Layer Equations

There are only a few *well-known* solutions to the boundary layer equations (mostly Blausius' solution and Falkner-Skan flows), but there are many others that have been developed. A quick look through Chapter 9 of Schlichting reveals the following two-dimensional solutions:

1. Flow past a wedge.
2. Flow in a convergent channel.
3. Flow past a cylinder; symmetrical case (Blausius series).
4. Boundary layer for the potential flow given by $V_e(x) = V_e - ax^n$.
5. Flow in the wake of flat plate at zero incidence.

6. The two-dimensional laminar jet.

7. Parallel streams in laminar flow.

8. Flow in the inlet length of a straight channel.

Most of these solutions are based on similarity methods: the transformation of the coordinates of the problem so that the solution in the new coordinates collapses onto an universal solution.

A variety of other three-dimensional and two-dimensional compressible solutions also exist. Look through Schlichting for more information. Although all of these solutions were obtained in incredibly elegant ways, for all but the simplest of geometries, the solutions tend to have serious accuracy shortcomings.

Laminar Boundary Layer Along a Flat Plate

The boundary layer equations can be rewritten for a zero pressure gradient situation as follows (introducing the coefficient of kinematic viscosity, ν):

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{3}$$

which are to be solved subject to the following boundary conditions:

$$\text{at } y = 0: \quad u = v = 0$$

$$\text{at } y = \infty: \quad u = V_e$$

We have two equations and two unknowns (u, v). The typical approach is to combine these two equations into a single equation in one unknown. This is easily accomplished using the concept of a *stream-function* that automatically satisfies the continuity equation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

If we were to substitute these expressions into the x-momentum boundary layer equation, we would quickly realize that the resulting ordinary differential equation for ψ is nonlinear on this variable. Instead, we try to express the function $\psi(x, y) = \psi(\eta)$, where η is a function of x and y itself. Blausius found that if the following choices were made

$$\eta = \frac{y}{2}\left(\frac{V_e}{\nu x}\right)^{\frac{1}{2}} \quad ; \qquad \psi = (\nu V_e x)^{\frac{1}{2}} f(\eta)$$

the resulting equation is the following ordinary differential equation

$$f''' + ff'' = 0 \tag{4}$$

where the transformed boundary conditions become

$$\begin{aligned} \text{at } \eta = 0: &\quad f = f' = 0 \\ \text{at } \eta = \infty: &\quad f' = 2 \end{aligned} \tag{5}$$

which can be shown to correspond to the no-slip boundary condition and the velocity matching boundary condition at the edge of the boundary layer.

The usual way in which this equation can be solved is called Weyl's method which assumes a series expansion on powers of η of $f(\eta)$ about $\eta = 0$.

$$f(\eta) = A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \cdots + \frac{A_n}{n!}\eta^n$$

The first two coefficients are eliminated by using the no-slip boundary condition, $A_0 = A_1 = 0$. The remaining terms of the series can be substituted into the governing Equation 4, and, after multiplying the various derivatives of f and collecting the terms on equal powers of η, we obtain

$$A_3 + A_4\eta + \left(\frac{A_2^2}{2!} + \frac{A_5}{2!}\right)\eta^2 + \cdots = 0$$

Considering that the equation must hold for all values of η, then, $A_3 = A_4 = 0$ and $A_2^2 + A_5 = 0$, and the solution can be written in terms of A_2 alone. The boundary condition at the edge of the boundary layer can be used to determine the value of this parameter.

The solution for f and its derivative obtained with this method can be seen in the Figure below.

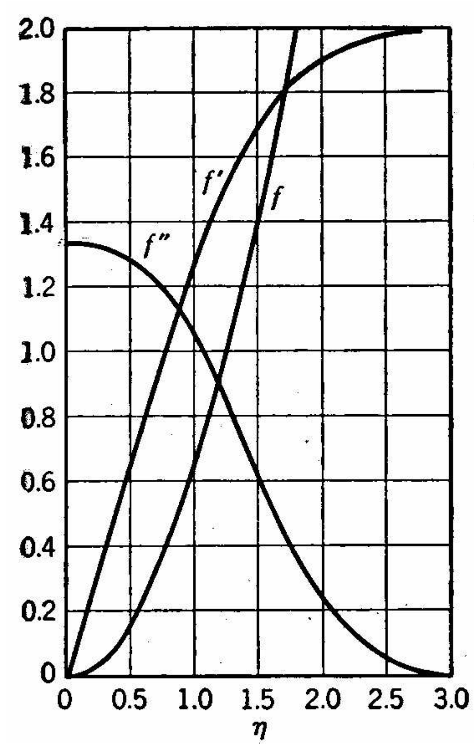


Figure 1: Blausius Solution for the Boundary Layer Equations

Most important are the results that can be derived from the solution of this equation. In particular, the value of u never reaches V_e (only at ∞).

however, at $\eta = 2.6, u/V_e = 0.994$. Therefore, if we choose the edge of the boundary layer ($y = \delta$) as the point where u is within 1 % of the free stream value, we obtain

$$\delta = 5.2\sqrt{\frac{\nu x}{V_e}} = \frac{5.2x}{\sqrt{Re_x}} \tag{6}$$

where $Re_x = V_e x/\nu$. Other important results can be derived from this solution. In particular, the *displacement thickness*, δ^*, whose definition we will discuss in a few slides, can be found to be

$$\delta^* = \frac{1.7208x}{\sqrt{Re_x}}$$

and also,

$$\tau_w = \frac{1}{4}\mu A_2 V_e \left(\frac{V_e}{\nu x}\right)^{\frac{1}{2}}$$

and therefore, the local coefficient of skin friction can be expressed as

$$c_f = \frac{A_2}{2}\left(\frac{\nu}{V_e x}\right)^{\frac{1}{2}} = \frac{0.664}{\sqrt{Re_x}}$$

When integrated along one side of the the length of the flat plate, l, we obtain

$$C_f = \int_0^l \frac{\tau_w dx}{\frac{1}{2}\rho V_e^2 l} = \frac{1.328}{\sqrt{Re_l}}$$

where $Re_l = V_e l/\nu$.

This seemingly simple result shows remarkable agreement with experiment in both velocity profiles and skin friction coefficient for large Re as long as the boundary layer remains laminar. These two facts can be seen from the Figures below. Unfortunately, for Reynolds numbers around 2×10^5 and above, the boundary layer becomes unstable and the flow *transitions* from *laminar* to *turbulent*. The skin friction increases significantly and the laminar results are no longer valid for either predictions of the skin friction coefficient or the actual velocity profiles. More on this later.

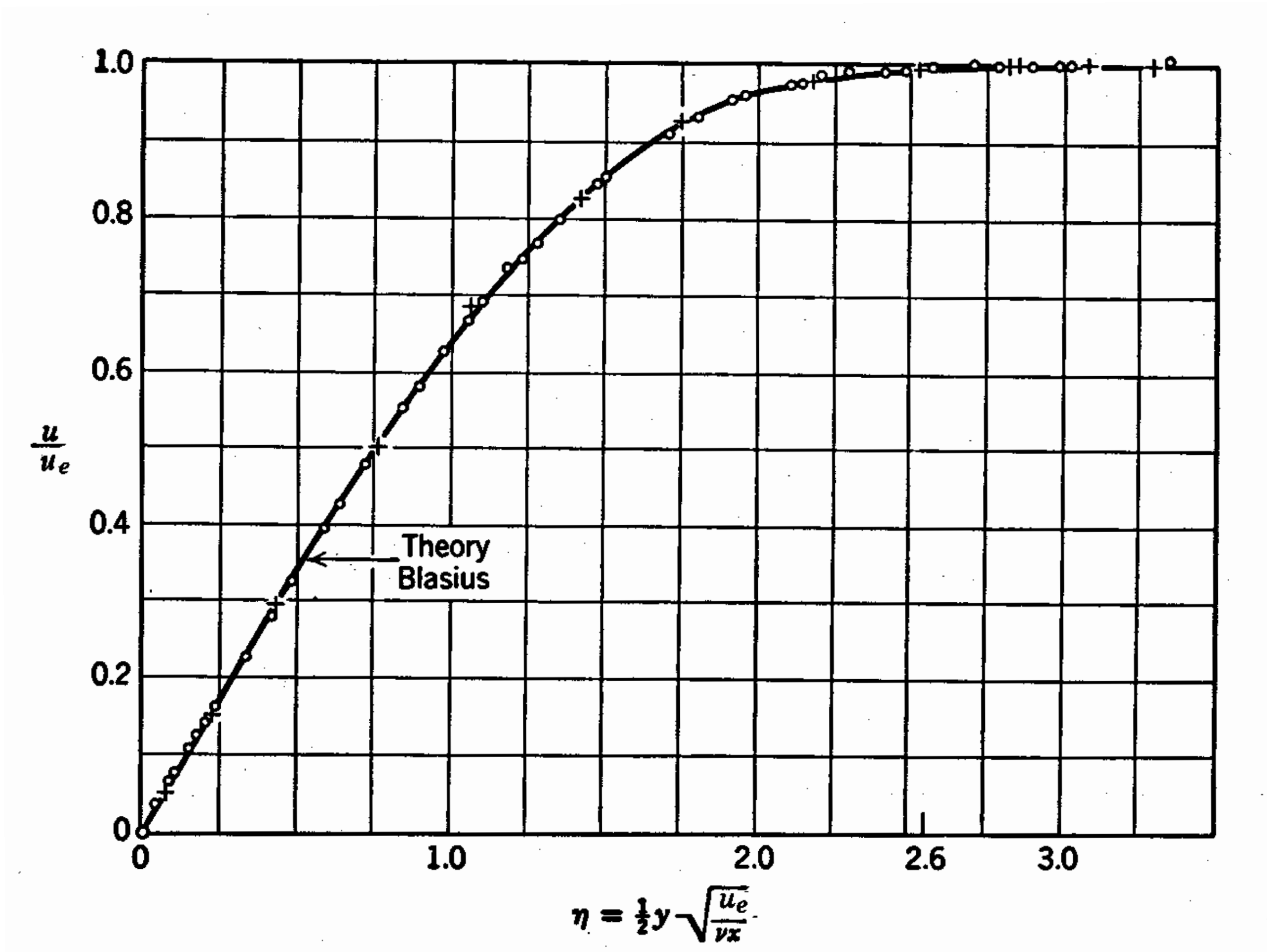


Figure 2: Experimental Comparison for Tangential Velocity Distribution

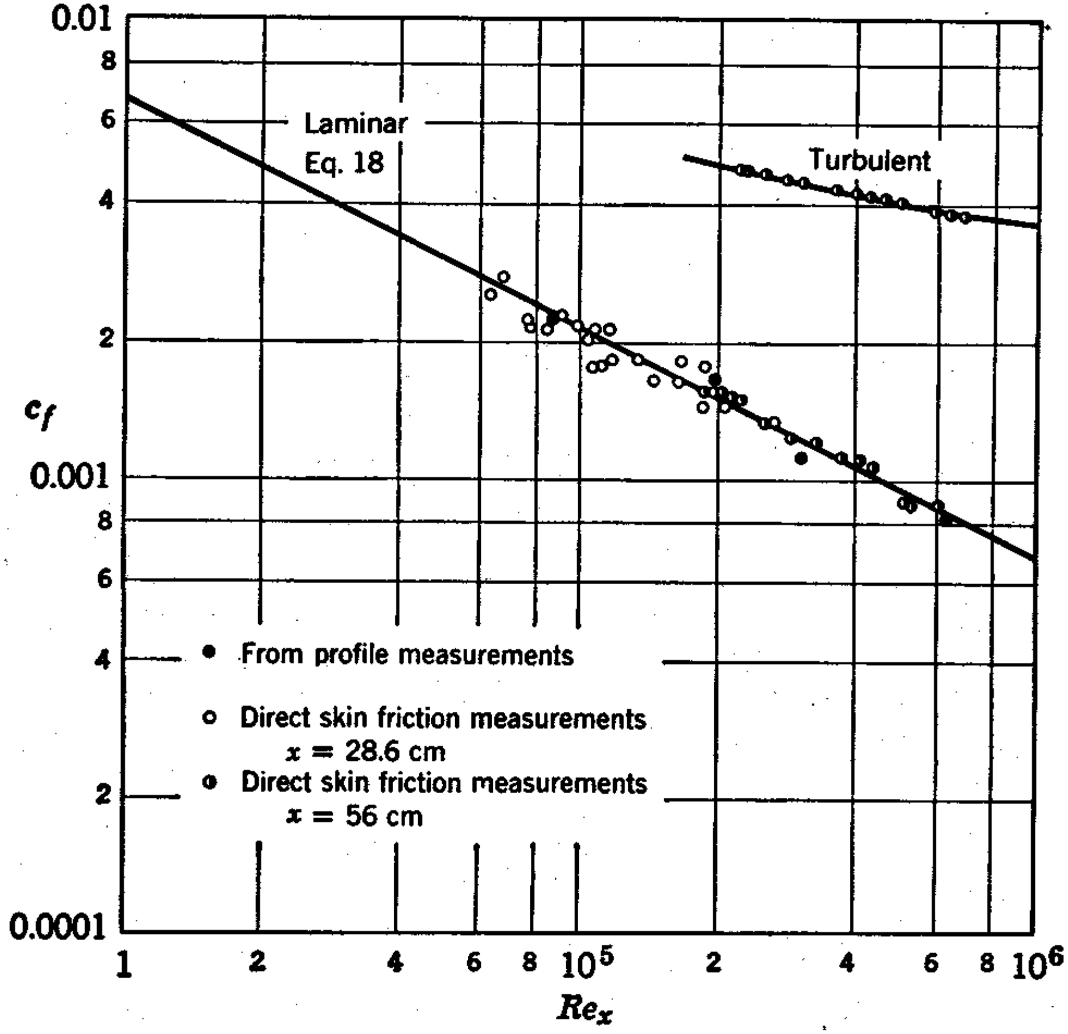


Figure 3: Local Skin Friction Coefficient for Incompressible Flow

Boundary Layer Displacement Thickness, δ^*

The definition of the boundary layer thickness, δ, is somewhat arbitrary. We can define δ as the distance from the wall at which the tangential velocity component reaches 99% of its free stream value. We can also find alternate definitions. Referring to

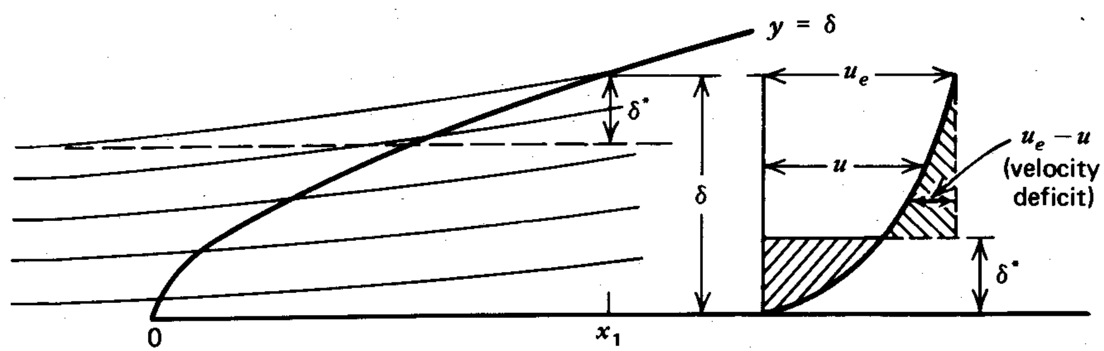


Figure 4: Boundary Layer Displacement Thickness, δ^*

Figure 4 above, we can see that δ^* at $x = x_1$ is the amount by which the streamline entering the boundary layer at that point has been displaced outward by the retardation of the flow in the boundary layer. The velocity profile shown at the right illustrates that, since the two cross-hatched areas are equal, the displacement thickness is given by the integral

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{V_e}\right) dy \tag{7}$$

An alternative explanation of the meaning of the displacement thickness can be found by looking at integrated volume fluxes at a given station along the length of the boundary layer. δ^* is the distance away from the wall that we would effectively have to *block* so that an outer flow with constant velocity V_e would allow the same mass flux as the actual boundary layer distribution integrated from the wall.

The additional importance of δ^* is that its evolution in the tangential direction can be used to impose a normal-velocity boundary condition on the outer inviscid flow. Take the definition of the normal velocity component inside the boundary layer

$$v(x,y) = \int_0^y \frac{\partial v}{\partial y}(x,y^*)dy^*$$

which, using the continuity equation we can say that

$$v(x,y) = -\int_0^y \frac{\partial u}{\partial x}(x,y^*)dy^* = -\frac{\partial}{\partial x}\int_0^y u dy^*$$

which can be written as

$$v(x,y) = \frac{\partial}{\partial x}\int_0^y \left[V_e(x) - u(x,y^*)\right] dy^* - y\frac{dV_e}{dx}$$

If y^* is large compared to δ, $u(x,y^*) \approx V_e(x)$ and the upper bound of the integral can be extended to ∞ without changing the value of the integral.

Then, this expression can be re-written as

$$v(x,y) = \frac{d}{dx}V_e\delta^* - y\frac{dV_e}{dx}$$

Finally, if we particularize this equation at the wall $(y = 0)$ we see that

$$V_n|_{y/L=0} = \frac{d}{dx}(V_e\delta^*) \quad (8)$$

From this derivation, the following crude iterative procedure can be followed to obtain a coupled viscous-inviscid solution.

1. Analyze the flow as if it were inviscid and compute the tangential velocity component $V_e(x)$ that results from this inviscid analysis.

2. Using the $V_e(x)$ distribution, use the boundary layer equations to calculate its evolution and to compute $\delta^*(x)$.

3. Reanalyze the inviscid flow subject to the modified flow tangency boundary condition given by Equation 8.

4. If the variation in $\delta^*(x)$ is large, go to 1.

5. Otherwise, the flow has converged and the solution is the correct one.

Notice that this procedure is effectively equivalent to re-analyzing the outer inviscid flow with the displacement *effect* of the boundary layer accounted for.

However, the modified flow tangency boundary condition is imposed at the original airfoil surface, and not at the actual edge of the boundary layer.

This procedure, although more accurate would require the re-panelization of the surface geometry which would now extend to infinity (wake region). The Figure below shows how this simple boundary layer correction can make a substantial difference in some situations and bring the computed results much closer to experiment than the original inviscid flow.

Von Kármán Momentum Integral Equation

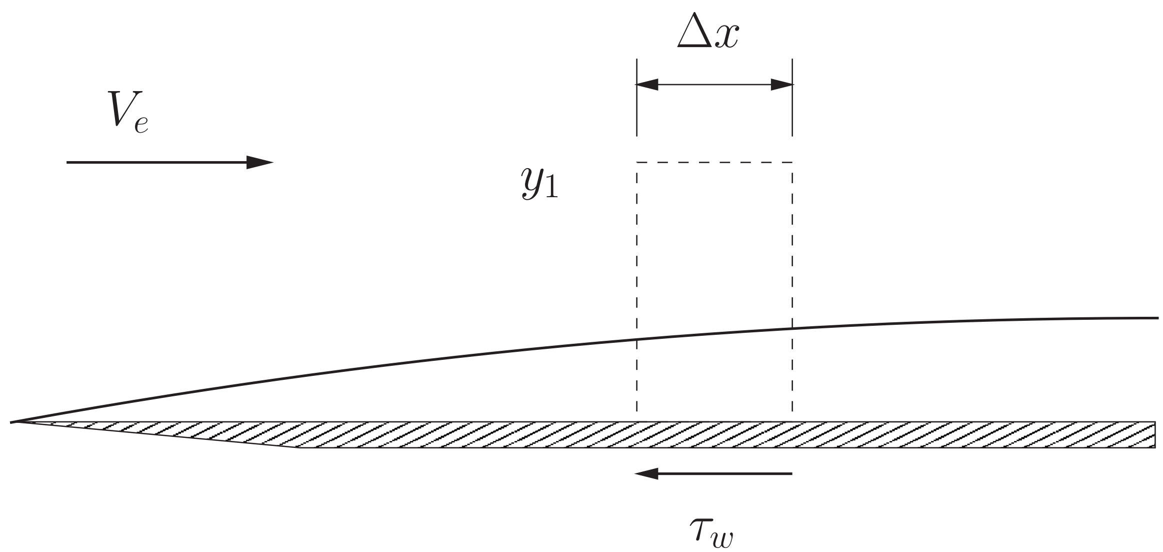


Figure 5: Notation for the Derivation of the Integral Momentum Equation

An alternative form of the boundary layer equations can be derived in integral form from the integral conservation of momentum statement. The following derivation is different from the one in your handout, but arrives at the same conclusion. All notation is based on the diagram in the Figure above.

In the control volume of the figure above, we are losing x-momentum trough the action of the shear stress at the wall, τ_w. The statement of conservation of momentum in integral form, for the control volume shown above can be written as follows:

$$\begin{aligned} - & \tau_w \Delta x + \int_0^{y_1} p dy - \int_0^{y_1} \left(p + \frac{\partial p}{\partial x} \Delta x \right) dy \\ = & -\int_0^{y_1} \rho u^2 dy + \int_0^{y_1} \left[\rho u^2 + \frac{\partial}{\partial x} (\rho u^2) \Delta x \right] dy + \rho_e v_{y_1} \Delta x V_e \end{aligned} \quad (9)$$

Dividing through by Δx, in the limit of $\Delta x \rightarrow 0$, using Bernoulli's equation to transform the term $\partial p / \partial x$, and using the integral form of the conservation of mass statement for the last term in Equation 9 we have:

$$-\tau_w + \int_0^{y_1} \rho_e V_e \frac{dV_e}{dx} dy = \int_0^{y_1} \frac{\partial}{\partial x}(\rho u^2) dy - V_e \int_0^{y_1} \frac{\partial}{\partial x}(\rho u) dy \qquad (10)$$

and expanding the last term

$$-\tau_w + \int_0^{y_1} \rho_e V_e \frac{dV_e}{dx} dy = \int_0^{y_1} \frac{\partial}{\partial x}(\rho u^2) dy - \int_0^{y_1} \frac{\partial}{\partial x}(\rho u V_e) dy + \int_0^{y_1} \rho u \frac{\partial V_e}{\partial x} dy \qquad (11)$$

Re-arranging terms

$$\tau_w = \int_0^{y_1} (\rho_e V_e - \rho u) \frac{\partial V_e}{\partial x}(\rho u^2) dy + \int_0^{y_1} \frac{\partial}{\partial x}(\rho u V_e - \rho u^2) dy \qquad (12)$$

and introducing the definition of the displacement thickness δ^*, we have

$$\tau_w = \frac{dV_e}{dx}\rho_e V_e \delta^* + \frac{d}{dx}\left[\rho_e V_e^2 \int_0^\infty \frac{\rho u}{\rho_e V_e}\left[1 - \frac{u}{V_e}\right] dy\right] \tag{13}$$

If we identify the integral with the momentum thickness of the boundary layer, θ,

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e V_e}\left[1 - \frac{u}{V_e}\right] dy \tag{14}$$

we obtain

$$\tau_w = \frac{dV_e}{dx}\rho_e V_e \delta^* + \frac{d}{dx}\left[\rho_e V_e^2 \theta\right] \tag{15}$$

which can be expanded to give

$$\tau_w = \rho_e V_e^2 \frac{d\theta}{dx} + 2\rho_e V_e \frac{dV_e}{dx}\theta + \rho_e V_e \frac{dV_e}{dx}\delta^* \tag{16}$$

which upon division by $\rho_e V_e^2$ yields the well-known *Von Kármán Momentum Integral Equation*

$$\frac{d\theta}{dx} + \frac{\theta}{V_e}(H+2)\frac{dV_e}{dx} = \frac{1}{2}c_f \tag{17}$$

where $H = \frac{\delta^*}{\theta}$ is the shape factor. This expression contains a useful relationship between the most important variables in boundary layer theory: δ^*, θ, and c_f. However, this equation contains far too many unknowns (θ, H, and c_f) to be useful by itself. It must be supplemented with additional information in the form of other equations. A variety of methods to provide this additional information exist and will be discussed soon.