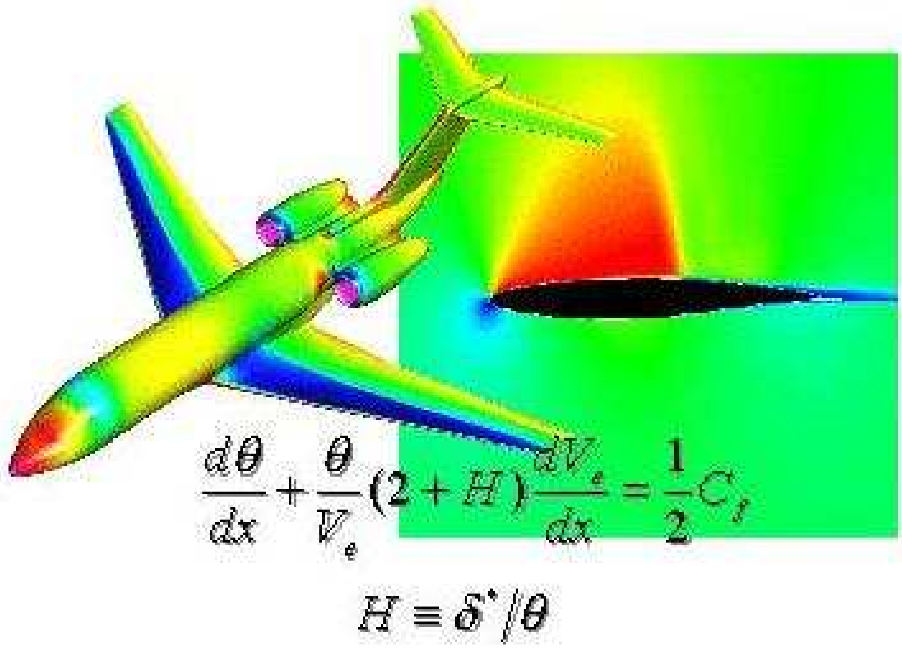
Solutions of the Von Kármán Integral Momentum Equation



AA200b

Lecture 8-9

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Von Kármán Momentum Integral Equation

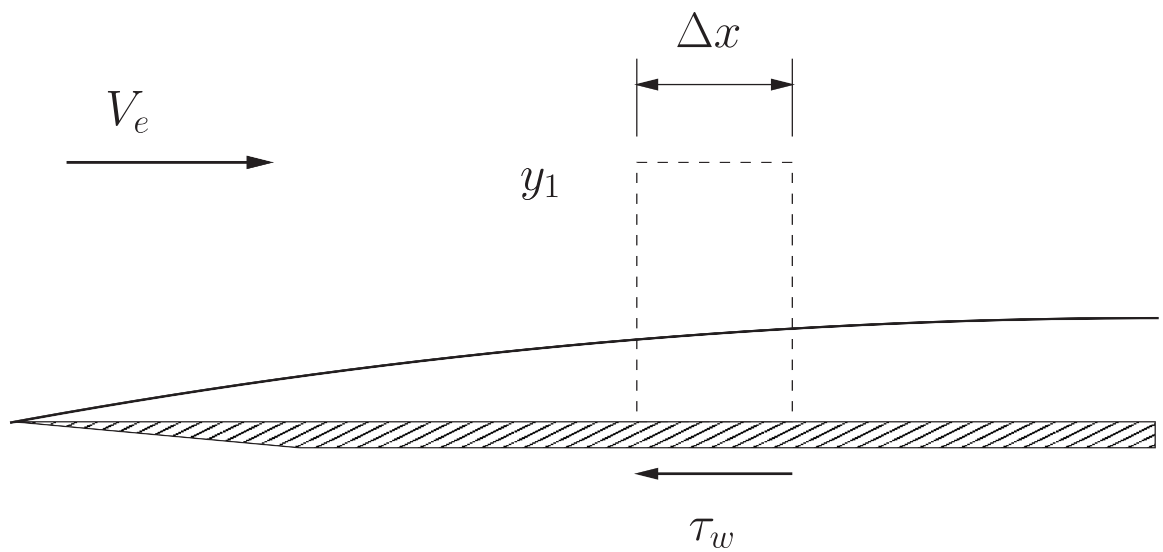

Figure 1: Notation for the Derivation of the Integral Momentum Equation

An alternative form of the boundary layer equations can be derived in integral form from the integral conservation of momentum statement. The following derivation is different from the one in your handout, but arrives at the exact same conclusion. All notation is based on the diagram in the Figure above.

Within the control volume of the figure above, we are losing x-momentum through the action of the shear stress at the wall, τ_w. The statement of conservation of momentum in integral form, for the control volume shown above can be written as follows ($\sum$ Forces on fluid $=$ Total x-momentum flux into the control volume):

$$\begin{aligned} - & \quad \tau_w \Delta x + \int_0^{y_1} p dy - \int_0^{y_1} \left(p + \frac{\partial p}{\partial x} \Delta x \right) dy \\ = & \quad - \int_0^{y_1} \rho u^2 dy + \int_0^{y_1} \left[\rho u^2 + \frac{\partial}{\partial x} (\rho u^2) \Delta x \right] dy + \rho v_{y_1} \Delta x V_e \qquad (1) \end{aligned}$$

Dividing through by Δx, in the limit of $\Delta x \to 0$, using Bernoulli's equation to transform the term $\partial p / \partial x$, and using the integral form of the conservation of mass statement for the last term in Equation 1 we have:

$$-\tau_w + \int_0^{y_1} \rho V_e \frac{dV_e}{dx} dy = \int_0^{y_1} \frac{\partial}{\partial x}(\rho u^2) dy - V_e \int_0^{y_1} \frac{\partial}{\partial x}(\rho u) dy \qquad (2)$$

and expanding the last term

$$-\tau_w + \int_0^{y_1} \rho V_e \frac{dV_e}{dx} dy = \int_0^{y_1} \frac{\partial}{\partial x}(\rho u^2) dy - \int_0^{y_1} \frac{\partial}{\partial x}(\rho u V_e) dy + \int_0^{y_1} \rho u \frac{\partial V_e}{\partial x} dy \qquad (3)$$

Re-arranging terms

$$\tau_w = \int_0^{y_1} (\rho V_e - \rho u) \frac{\partial V_e}{\partial x} dy + \int_0^{y_1} \frac{\partial}{\partial x}(\rho u V_e - \rho u^2) dy \qquad (4)$$

and introducing the definition of the displacement thickness δ^*, we have

$$\tau_w = \frac{dV_e}{dx}\rho V_e \delta^* + \frac{d}{dx}\left[\rho V_e^2 \int_0^\infty \frac{u}{V_e}\left[1 - \frac{u}{V_e}\right] dy\right] \tag{5}$$

If we identify the integral with the momentum thickness of the boundary layer, θ,

$$\theta = \int_0^\infty \frac{u}{V_e}\left[1 - \frac{u}{V_e}\right] dy \tag{6}$$

which is simply the distance from the wall for which a *jet* of velocity V_e would carry an amount of momentum equal to the momentum deficit created by the presence of the boundary layer, we would obtain

$$\tau_w = \frac{dV_e}{dx}\rho V_e \delta^* + \frac{d}{dx}\left[\rho V_e^2 \theta\right] \tag{7}$$

which can be expanded to give

$$\tau_w = \rho V_e^2 \frac{d\theta}{dx} + 2\rho V_e \frac{dV_e}{dx}\theta + \rho V_e \frac{dV_e}{dx}\delta^* \tag{8}$$

which upon division by ρV_e^2 yields the well-known *Von Kármán Momentum Integral Equation*

$$\frac{d\theta}{dx} + \frac{\theta}{V_e}(H+2)\frac{dV_e}{dx} = \frac{1}{2}c_f \tag{9}$$

where $H = \frac{\delta^*}{\theta}$ is the shape factor. This expression contains a useful relationship between the most important variables in boundary layer theory: δ^*, θ, and c_f. However, this equation contains far too many unknowns (θ, H, and c_f) to be useful by itself. It must be supplemented with additional information in the form of other equations. A variety of methods to provide this additional information exist and will be discussed soon.

Integral Results for Flat Plate Flow

In order to illustrate some of the properties of the integral momentum equation, let's take the example of the flat plate at zero incidence. Since the external pressure gradient is equal to zero, so will be the term dV_e/dx by using Bernoulli's equation for the outer inviscid flow. In that case, Equation 9 reduces to

$$c_f = 2\frac{d\theta}{dx} \tag{10}$$

which shows that the value of the local coefficient of friction is directly related to the local slope of the momentum thickness, θ. Moreover, notice that since the boundary layer is typically losing momentum (i.e. $c_f > 0$, then the boundary layer is growing, $\frac{d\theta}{dx} > 0$. Moreover, from results obtained by Blausius, we can arrive at estimates of both the momentum and the displacement thicknesses, and their ratio, the *shape factor*.

From previous results, we know that

$$c_f \sqrt{Re_x} = 0.664$$

and therefore

$$\frac{d\theta}{dx} = \frac{0.332}{\left(\frac{V_e}{\nu}\right)^{\frac{1}{2}}} x^{-1/2}$$

which can be integrated to give

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

Remember that the value of the original definition of the boundary layer thickness, $\delta_{0.99}$ was given by

$$\frac{\delta_{0.99}}{x} = \frac{5.2}{\sqrt{Re_x}}$$

and therefore,

$$\frac{\delta_{0.99}}{\theta} = 7.83$$

Similarly, the estimate of the displacement thickness, δ^* was given by

$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}}$$

and for the case of $\frac{dp}{dx}$ one finds that

$$\frac{\delta^*}{\theta} = 2.59$$

For fuller profiles we would expect $\delta^* \approx \theta$, while close to separation, typically, $\delta^* \gg \theta$. Herein lies the importance of the *shape factor*: it can be used as an indication of the tendency of the boundary layer toward stability, or towards separation. Given that

$$H = \frac{\delta^*}{\theta}$$

we can expect that

$$\begin{aligned} H &< 2.59 \quad \text{for} \quad \frac{dp}{dx} < 0 \\ H &> 2.59 \quad \text{for} \quad \frac{dp}{dx} > 0 \end{aligned}$$

The shape factor (which we will derive from the integral solutions to the boundary layer equations) is then an intuitive measure of how close we are to *separation*: the point at which the local c_f becomes zero, and the flow close to the wall reverses its direction.

In particular, the figures below show a typical solution for a NACA4410 airfoil at a Reynolds number, $Re_c = 500,000$ and at an angle of attack, $\alpha = 4^\circ$. Notice that the airflow needs to negotiate an adverse pressure gradient on the upper surface, which increases the value up until the point of separation (point at which $c_f = 0.0$) as expected from our simple explanation.

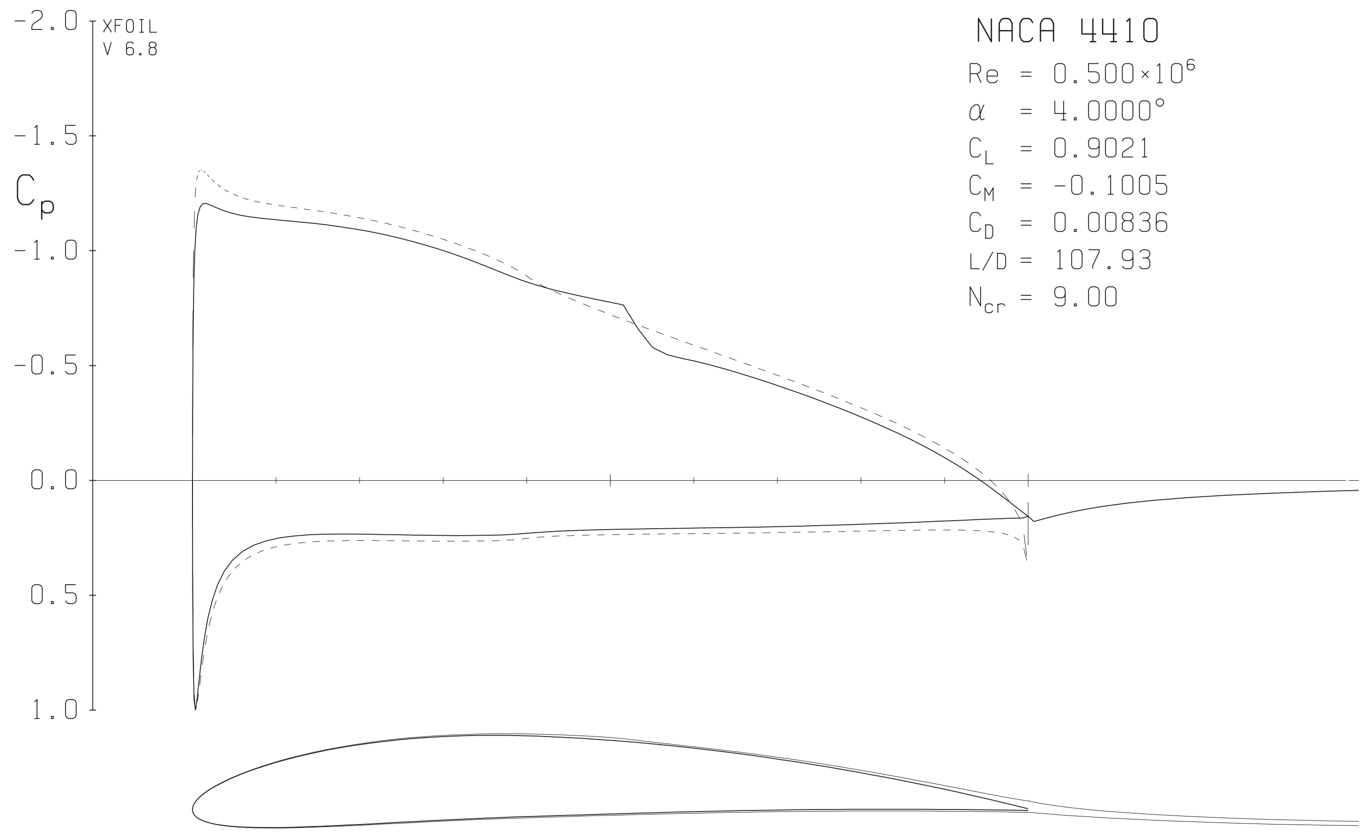

Figure 2: C_p Distribution for NACA 4410 Airfoil at $\alpha = 4^\circ$ and $Re_c = 500,000$.

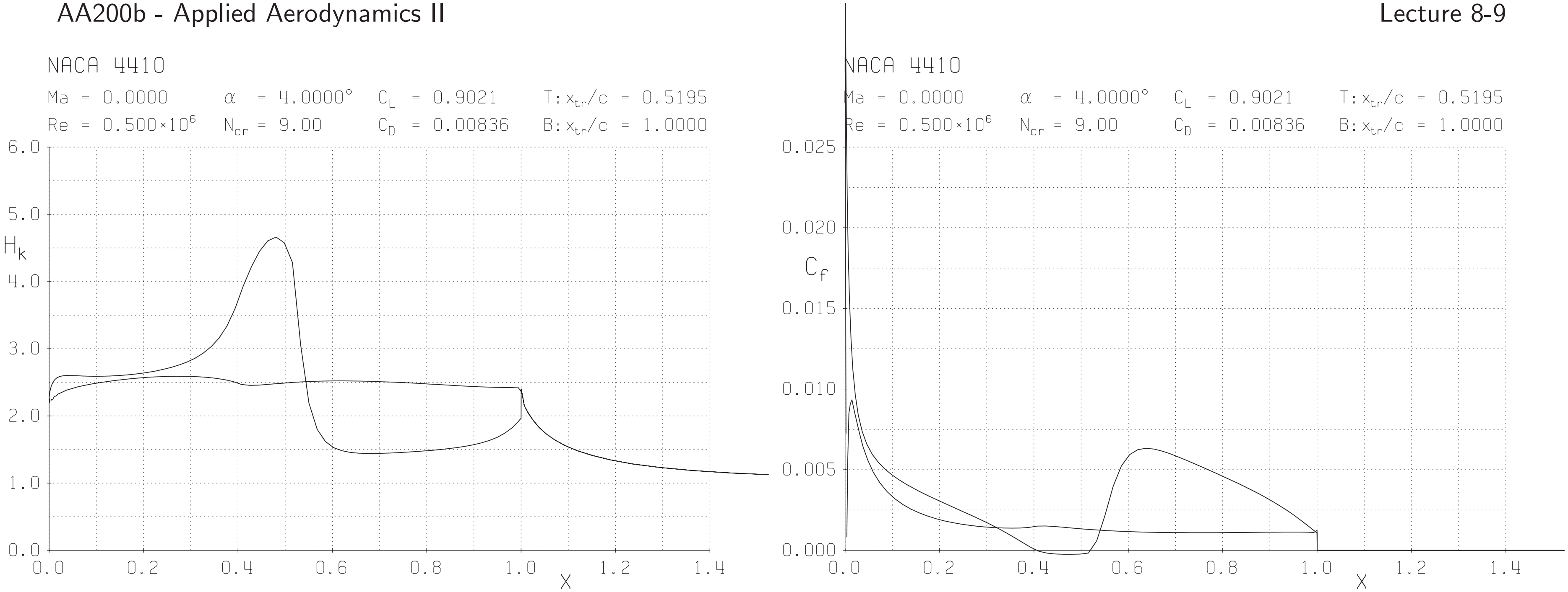

Figure 3: Shape Factor, H, and Local c_f for a NACA 4410 Airfoil at $\alpha = 4^\circ$ and $Re_c = 500,000$.

Numerical Solution of the Boundary Layer Equations in Integral Form

Today we will discuss the *numerical* solution of the boundary layer equations for non-similar flows (those for which u/V_e is a function of both x and η). In practice these flow are more important than similar flows because $V_e(x)$ rarely varies in such a way to make a similarity solution possible. Moreover, the surface boundary conditions may not satisfy the requirements for similarity, even if V_e does.

While the differential methods that we will briefly discuss next time are more general and accurate, integral methods are very useful in obtaining quick rough answers for some kinds of flows. Of the many integral methods for laminar flows, especially popular in the pre-computer era, we will discuss both the *Pohlhausen Method* (1921) because of its simplicity and the *Thwaites Method* which is very useful, and often still used to calculate the short portion of laminar flow in the leading edge area of high Reynolds number airfoil and wing flows before switching to a turbulent calculation.

In sum, notice that we have two numerical solution alternatives:

1. Solution of the Integral Equation given by Equation 9, or

2. Solution of the actual Boundary Layer Equations which we derived in a previous lecture.

In general, the first item is easier to accomplish (not without additional *guesses*), while the second one is more involved, but also much more applicable to realistic flows.

Pohlhausen Method - 1921

In this method, we assume a velocity profile $u(x, y)$ that satisfies the momentum integral equation, which is repeated below for completeness:

$$\frac{d\theta}{dx} + \frac{\theta}{V_e}(H+2)\frac{dV_e}{dx} = \frac{1}{2}c_f \tag{11}$$

and set the boundary conditions

$$u, v = 0, \quad y = 0, \quad \text{and} \quad u = V_e(x), \quad y \to \infty \tag{12}$$

We will also use *additional* boundary conditions obtained by evaluating Equation 11 at the wall with $v_w = 0$, that is,

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{dp}{dx} = -V_e \frac{dV_e}{dx}$$

and also some more boundary conditions obtained from differentiating the edge boundary condition with respect to y, namely,

$$\text{as} \quad y \to \infty, \quad \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial y^3}, \ldots \to 0$$

We will now assume a fourth-order polynomial variation of the velocity profile, u/V_e and write

$$\frac{u}{V_e} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \tag{13}$$

where η is now simply $\eta = y/\delta$. Notice that this polynomial approximation is consistent with the fact that the solution to the Blausius problem behaved linearly in the region *close* to the wall, while it behaved as a fourth order polynomial in the outer portions of the boundary layer.

The polynomial in Equation 13 contains five coefficients that can be determined from the boundary conditions outlined earlier. Using these boundary conditions, we can obtain the values of the coefficients as a function of a single parameter, Λ, since

$$a_0 = 0, \quad a_1 = 1 + \frac{\Lambda}{6}, \quad a_2 = -\frac{\Lambda}{2}, \quad a_3 = -2 + \frac{\Lambda}{2}, \quad a_4 = 1 - \frac{\Lambda}{6} \tag{14}$$

where Λ is a pressure gradient parameter defined by

$$\Lambda = \frac{\delta^2}{\nu}\frac{dV_e}{dx} \tag{15}$$

With these values of the coefficients, the polynomial approximation can be written as a function of Λ alone as follows

$$\frac{u}{V_e} = (2\eta - 2\eta^3 + \eta^4) + \frac{1}{6}\Lambda\eta(1-\eta)^3 \tag{16}$$

The figure below shows the family of velocity distributions that can be obtained by simply varying this parameter

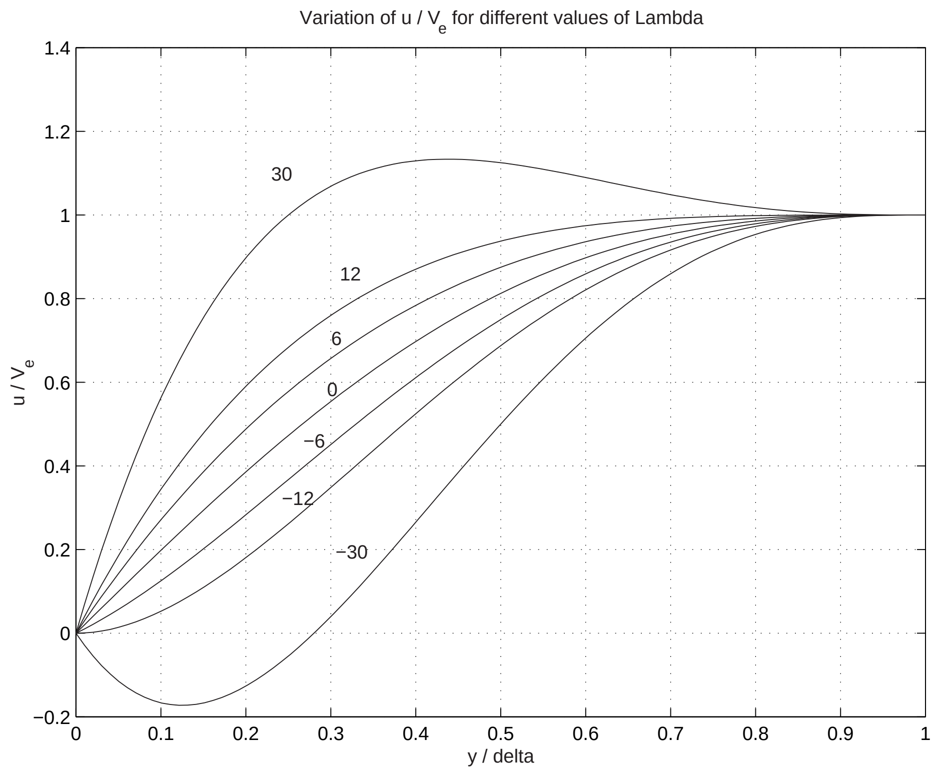

Figure 4: Family of Velocity Profiles obtained by Varying the Parameter Λ

Notice that $\Lambda = 0$ corresponds to a flat-plate flow. A negative value of Λ corresponds to a decelerating flow, and a positive value of Λ corresponds to an accelerating flow. The profile at separation $c_f = 0$ corresponds to $\Lambda = -12$, since it yields $\frac{\partial u}{\partial y} = 0$ at $\eta = y = 0$. The positive values of Λ are restricted to 12 since otherwise the velocity profile overshoots the value of the edge velocity, V_e. Therefore

$$-12 \leq \Lambda \leq 12$$

Knowing the velocity profile as a function of Λ, we can easily find that

$$\begin{aligned} \tau_w &= \frac{\mu V_e}{\delta}\left(2 + \frac{1}{6}\Lambda\right) \\ \delta^* &= \delta\left(\frac{3}{10} - \frac{1}{120}\Lambda\right) \end{aligned}$$

$$\theta \;=\; \frac{\delta}{315}\left(37 - \frac{1}{3}\Lambda - \frac{5}{144}\Lambda^2\right)$$

and substituting into Equation 11 we obtain the following ordinary differential equation

$$\frac{dZ}{dx} = \frac{g(\Lambda)}{V_e} + h(\Lambda)Z^2\frac{d^2V_e}{dx^2} \tag{17}$$

where $Z = \delta^2/\nu = \Lambda/(du_e/dx)$ and both $g(\Lambda)$ and $h(\lambda)$ are known functions of Λ. This equation can be solved by your favorite numerical method. You can even use MATLAB to do the job for you.

Note that the auxiliary relations for c_f and H are not differential equations. Also note that only the initial values of δ or θ and the distribution of $V_e(x)$ are necessary to start the calculation.

Before computers were widely available, the Pohlhausen Method was the most sophisticated one in general use because the solution of the differential boundary layer equations was truly impracticable.

Thwaites Method

Consider the momentum integral Equation 11. If H and c_f are known functions of θ or some suitable combination of θ and V_e, Equation 11 can be easily integrated, at least by a numerical process. Such functions were found in Thwaites' method by writing the following boundary conditions

$$y = 0, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{V_e}{\theta^2}\lambda, \quad \frac{\partial u}{\partial y} = \frac{V_e}{\theta}l$$

These equations define λ and l. The variable l may be calculated by any particular solution of the boundary layer equations, and it is found in all known cases to adhere reasonably closely to a universal function of λ, which Thwaites denoted by $l(\lambda)$. In the same way, if H is regarded as depending only on λ, a reasonably valid universal function for H can also be found, namely $H(\lambda)$. By putting $y = 0$ in Equation 11 and using the definitions for λ and l above, we find

$$\lambda = \frac{\theta^2}{\nu}\frac{dV_e}{dx}$$

Also,

$$\frac{c_f}{2} = \frac{\tau_w}{\rho V_e^2} = \frac{\nu}{V_e^2}\left(\frac{\partial u}{\partial y}\right)_w = \frac{\nu l(\lambda)}{V_e \theta} = \frac{l}{Re_\theta}$$

The assumptions that l or c_f and H are functions of λ alone are quasi-similarity assumptions. The solutions to the Falkner-Skan problem can be used to give $l(\lambda)$ and $H(\lambda)$. With these two results, Equation 11 can be re-written in the form

$$\frac{V_e}{\nu}\frac{d\theta^2}{dx} = 2\left(-[H(\lambda)+2]\lambda + l(\lambda)\right) \equiv F(\lambda) \qquad (18)$$

Here $F(\lambda)$ is another universal function, which Thwaites chose to fit known solutions of the boundary layer equations with the function

$$F(\lambda) = 0.45 - 6\lambda = 0.45 - 6\frac{\theta^2}{\nu}\frac{\partial V_e}{dx} \tag{19}$$

which can now be substituted into Equation 18 to yield (after multiplying by V_e^5

$$\frac{1}{\nu}\frac{d}{dx}(\theta^2 V_e^6) = 0.45 V_e^5 \tag{20}$$

which can be integrated to give

$$\frac{\theta^2 V_e^6}{\nu} = 0.45\int_0^x V_e^5 dx + \left(\theta^2\frac{V_e^6}{\nu}\right)_0 \tag{21}$$

In terms of the dimensionless quantities defined by

$$x^* \equiv \frac{x}{L}, \quad u^* \equiv \frac{u}{u_{ref}}, \quad V_e^* \equiv \frac{V_e}{u_{ref}}, \quad Re_L = \frac{u_{ref}L}{\nu}$$

Equation 21 can be written as

$$\left(\frac{\theta}{L}\right)^2 Re_L = \frac{0.45}{(V_e^*)^6}\int_0^{x^*}(V_e^*)^5 dx^* + \left(\frac{\theta}{L}\right)_0^2 Re_L \left(\frac{V_{e0}^*}{V_e^*}\right)^6 \qquad (22)$$

For a stagnation point ($m = 0$ in the Falkner-Skan formulation), this Equation can be simplified to

$$\left(\frac{\theta}{L}\right)^2 Re_L = \frac{0.075}{\left(\frac{dV_e^*}{dx^*}\right)_0}$$

where dV_e^*/dx^* denotes the slope of the external velocity distribution for a stagnation-point flow. Note that the last term in Equation 22 is zero in calculations starting from a stagnation point, because $V_e^* = 0$ at that point.

Once θ is calculated from a given external velocity distribution, the other boundary layer parameters, H, and c_f can be determined from the relations given below.

For $0 \leq \lambda \leq 0.1$,

$$\begin{aligned} l &= 0.22 + 1.57\lambda - 1.8\lambda^2 \\ H &= 2.61 - 3.75\lambda + 5.24\lambda^2 \end{aligned}$$

For $-0.1 \leq \lambda \leq 0$,

$$l = 0.22 + 1.402\lambda + \frac{0.018\lambda}{0.107 + \lambda}$$

$$H = \frac{0.0731}{0.14+\lambda} + 2.088$$