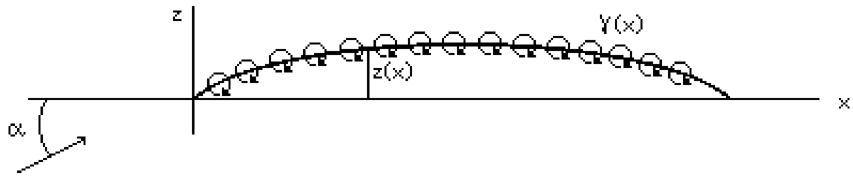
5.7.2.1. Thin Airfoil Theory Derivation

We start with the analysis of a very thin cambered plate and will build up the solution to a more arbitrary airfoil.

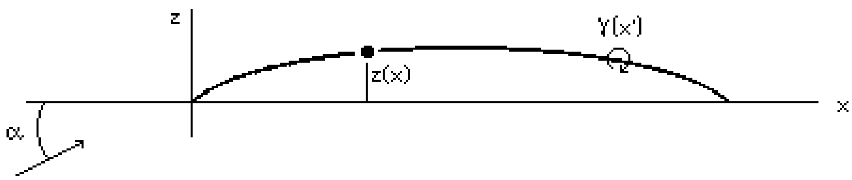


A distribution of vorticity on the airfoil will be a solution to Laplace's equation. It will satisfy the boundary conditions if the combination of the velocity induced by the vortices cancels the component of the freestream normal to the plate:

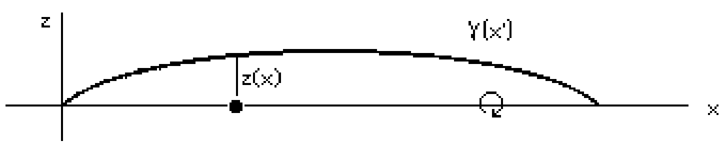
$$w_i(x) = U_\infty \left(\alpha - \frac{dz}{dx} \right)$$

(where small angle approximations have been introduced)

The basic approximation of thin airfoil theory is that the velocity induced at some point x due to the vorticity at x'...



... may be approximated by the velocity induced at the same x position on the x axis due to a vortex on the x axis:



The velocity induced by this bit of vorticity is computed from the basic vortex singularity. The formula is known as the Biot-Savart Law and in 2-D for the element of vorticity at x', it reads:

$$dw(x) = \frac{\gamma(x')\, dx'}{2\pi (x-x')}$$

So, the total induced velocity at the point x is given by:

$$w(x) = \frac{1}{2\pi} \int_0^1 \frac{\gamma(x')\, dx'}{x-x'} \qquad (1)$$

Combining this expression with the flow-tangency boundary condition, we have the basic integral equation to be solved for the unknown vorticity distribution:

$$\frac{1}{2\pi U_\infty} \int_0^1 \frac{\gamma\, dx'}{x-x'} = \alpha - \frac{dz(x)}{dx} \qquad (2)$$

The approach to solving this equation is to change variables:

$$\cos\theta = 1 - 2x \quad (\theta \text{ varies from } 0 \text{ to } \pi) \qquad (3)$$

and to write γ as a Fourier series:

$$\gamma(\theta) = 2 U_\infty [A_0 \cot\frac{\theta}{2} + \sum_1^\infty A_n \sin n\theta] \qquad (4)$$

Substituting (4) into (2) and using the trigonometric relations:

$$[\cos(n-1)x - \cos(n+1)x] = 2 \sin x \sin nx$$

$$\int_0^\pi \frac{\cos n\theta'}{\cos\theta' - \cos\theta} d\theta' = \pi \frac{\sin n\theta}{\sin\theta}$$

yields:

$$A_0 - \sum_1^\infty A_n \cos n\theta = \alpha - \frac{dz}{dx} \qquad (5)$$

Finally, we multiply both sides by cos mθ and integrate from 0 to π:

$$\boxed{\begin{aligned} A_0 &= \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta \\ A_n &= \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta \, d\theta \end{aligned}} \qquad (6)$$

We can substitute these coefficients into the expression for γ (4), to find the vorticity, pressures, lift, and moment as a function of the surface slope distribution, dz/dx.

In particular, the expressions for the local pressure difference and integrated lift and moment about the leading edge are:

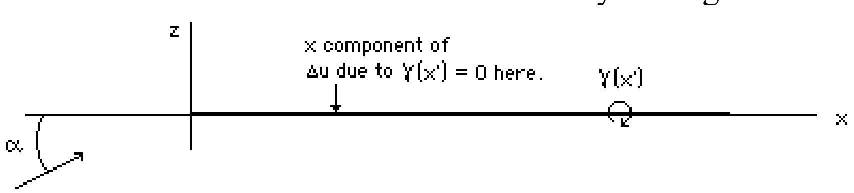
$$C_{p_l} - C_{p_u} = \frac{\rho U_\infty \gamma}{\rho U_\infty^2 / 2} = \frac{2\gamma}{U_\infty} = 4 \ [A_0 \cot\frac{\theta}{2} + \sum_1^\infty A_n \sin n\theta]$$

$$l = \int_0^1 \rho U_\infty \gamma \, dx \qquad \text{or} \qquad C_l = 2\pi (A_0 + \frac{A_1}{2})$$

$$m_{l.e.} = \int_0^1 \rho U_\infty \gamma \, x \, dx \qquad \text{or} \qquad C_m = -\frac{\pi}{2}(A_0 + A_1 - \frac{A_2}{2}) = -\frac{C_l}{4} - \frac{\pi}{4}(A_1 - A_2)$$

(Note that the expression relating C_p and γ applies only to thin airfoils.)

This method of computing the circulation distribution permits us to compute the pressure *difference* across the airfoil. The pressures on the upper and lower surfaces may also be computed by noting that at the surface the perturbation velocities in the x direction caused by the singularities are zero except for those due to the local vorticity:



The only contribution to du(x) is from the local vorticity (at x). It can be shown that this perturbation velocity is du = $\pm \gamma / 2$, with the + sign for the upper surface and the - sign for the lower surface.

5.7.2.2. Thin Airfoil Theory Results

The basic equations derived from thin airfoil theory are repeated below:

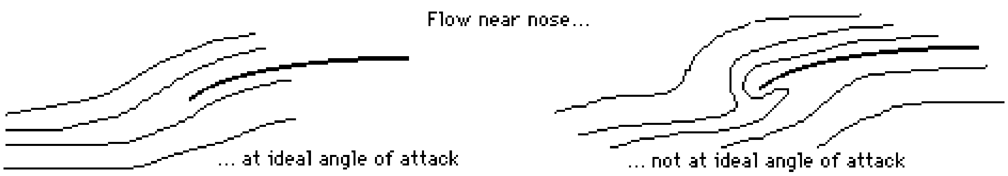
$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\,d\theta \qquad A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta\,d\theta$$

$$C_l = 2\pi\left(A_0 + \frac{A_1}{2}\right) \qquad C_m = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right) = -\frac{C_l}{4} - \frac{\pi}{4}(A_1 - A_2)$$

Several important results are derived from these expressions and are described in the following sections.

Ideal angle of attack

The constants A_n just depend on the airfoil shape -- except for A_0 which depends also on the angle of attack. The A_0 term is a strange term in the Fourier series for γ since it leads to a singularity at the leading edge (γ-> infinity). Thus, there is one angle of attack, called the ideal angle of attack, at which $A_0 = 0$ and the vorticity goes to 0 at the leading edge. This angle is called the ideal angle of attack.



Of course in real flows, the vorticity would not become infinite (why?), but the concept of ideal angle of attack is still important, identifying the flow conditions for which leading edge pressure peaks are avoided.

Lift curve slope

The rate of change of lift coefficient with angle of attack, $dC_L/d\alpha$ can be inferred from the expressions above.

The result, that C_L changes by 2π per radian change of angle of attack (.1096/deg) is not far from the measured slope for many airfoils. The effects of thickness and viscosity which are ignored here cancel each other out to some extent with the result that most airfoils have a lift curve slope within 10% of the 2π value given by thin airfoil theory.

Aerodynamic center and pitching moment at L = 0

The expression for pitching moment coefficient measured about the leading edge is given above. If we measure the moment about another reference center at a position x_0/c, the expression becomes:

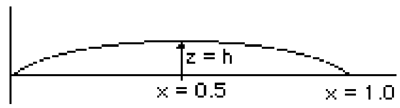
$$C_m = -\frac{C_l}{4} - \frac{\pi}{4}(A_1 - A_2) + C_l\,\frac{x_0}{c}$$

Note that if we choose the point $x_0 = 0.25c$, then the lift dependence drops out and the moment coefficient measured about this point is independent of the angle of attack*. The point about which $dC_m/dC_L = 0$ is called the aerodynamic center and according to thin airfoil theory it is the quarter chord point of the airfoil. Experiments show this to be quite close.

Results for a general example + Rules of Thumb

The parabolic camber meanline is used as an example of thin airfoil theory. Results for this case serve as useful first approximations for any thin cambered airfoil.

Assume: $z(x) = 4\,h\,x\,(1-x)$



$$\frac{dz}{dx} = 4h(1-2x) = 4h\cos\theta$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} 4h\cos\theta\, d\theta = \alpha$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} 4h\cos^2\theta\, d\theta = 4h$$

$$A_2 = 0$$

So, from the general formulas for Cl and Cm:

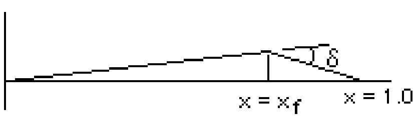
$$\boxed{C_l = 2\pi\alpha + 4\pi h = 2\pi(\alpha+2h) = 2\pi\alpha_{3c/4}}$$

(Note that $\alpha_{3c/4}$ is the angle of attack measured with respect to a line tangent to the airfoil at the 3/4 chord point.)

$$\boxed{\alpha_{ideal} = 0}$$

$$\boxed{C_{m_0} = -\pi h}$$

Thin airfoil theory can also be used to estimate the effect of flap deflection on airfoil lift and moment. It also provides an estimate of the hinge moments vs. the deflection angle and the angle of attack. This is a good problem to work on your own.



The results are:

$$\Delta C_l = [\,2(\pi-\theta) + 2\sin\theta\,]\,\delta$$

$$\Delta C_{m_o} = \frac{\delta}{2}\sin\theta\cos(\theta-1)$$

where: $x_f = \frac{1}{2}(1-\cos\theta)$

Because of the effects of viscosity, these results tend to overestimate, to some extent, the lift and moment due to flap deflections.

---

*The C_m about this point, denoted $C_{m_{a.c.}}$, is therefore equal to the C_m at zero lift. C_{m_0}, as it is written, is also independent of the moment reference location and so is a particularly useful quantity:
$C_{m_0} = C_{m_{a.c.}}$

5.7.2.3. Thin Airfoil Inverse Design

The basic thin airfoil theory formulation can be used to design airfoils with a desired pressure distribution. This process is actually easier than the direct analysis. The integral equation used to obtain the circulation distribution given the airfoil shape:

$$\frac{1}{2\pi U_\infty} \int_0^1 \frac{\gamma(x')\, dx'}{x-x'} = \alpha - \frac{dz(x)}{dx}$$

can be evaluated directly if we know the circulation distribution.

We evaluate the integral at several values of x to find the surface slope dz/dx along the airfoil.

$$\frac{dz(x)}{dx} = \alpha - f(x)$$

which can be integrated to obtain:

$$z(x) = \alpha\, x - \int_{x'=0}^{x} f(x')\, dx' + C$$

C and can be chosen to make z(0) = z(1) = 0.