1 Summary of ADI Scheme

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Discretize

$$\frac{\partial w}{\partial t} + R(w) = 0, \quad R(w) = \frac{\partial}{\partial x} f(w) + \frac{\partial}{\partial y} g(w) \tag{1.1}$$

by the trapezoidal rule

$$\begin{aligned} \frac{w^{n+1} - w^n}{\Delta t} &+ \frac{1}{2}\left(D_x f(w^{n+1}) + D_y g(w^{n+1})\right) \\ &+ \frac{1}{2}\left(D_x f(w^n) + D_y g(w^n)\right) \\ &= \frac{\partial w}{\partial t} + R(w) + \mathcal{O}(\Delta t^2). \end{aligned}$$

Linearize this by setting

$$\begin{aligned} f(w^{n+1}) &= f(w^n) + A\Delta w + \mathcal{O}(||\Delta w^2||) \\ g(w^{n+1}) &= g(w^n) + B\Delta w + \mathcal{O}(||\Delta w^2||), \end{aligned}$$

where $A = \frac{\partial f}{\partial w}$, $B = \frac{\partial g}{\partial w}$. Hence,

$$\frac{1}{\Delta t}\left\{I + \frac{1}{2}\Delta t(D_x A + D_y B)\right\}\Delta w + R(w^n) = \frac{\partial w}{\partial t} + R(w) + \mathcal{O}(\Delta t^2).$$

Use approximate factorization to reduce the complexity of the linearized scheme:

$$\frac{1}{\Delta t}\left\{I + \frac{1}{2}\Delta t D_x A\right\}\left\{I + \frac{1}{2}\Delta t D_y B\right\}\Delta w + R(w^n) = \frac{\partial w}{\partial t} + R(w) + \mathcal{O}(\Delta t^2),$$

where there is an additional factorization error

$$\frac{1}{4}\Delta t^2 D_x A D_y B \Delta w.$$

Problems:

1. The factorization error dominates at large CFL numbers
2. The scheme isn't amenable to parallel processing

2 ADI scheme with 3-4-1 Backward Euler

2.1 General Derivation

Discretize

$$\frac{\partial w}{\partial t} + R(w) = 0, \quad R(w) = \frac{\partial}{\partial x} f(w) + \frac{\partial}{\partial y} g(w) \tag{2.1}$$

by the fully implicit scheme

$$\frac{3}{2\Delta t} w^{n+1} - \frac{2}{\Delta t} w^n + \frac{1}{2\Delta t} w^{n-1} + R(w^{n+1}) = 0. \tag{2.2}$$

This is second order accurate and A-stable (the third order accurate scheme is stiffly stable). Using an inner iteration, solve for a general body fitted moving coordinate system with coordinates X_i with determinant of the transformation defined as

$$J = \left| \frac{\partial x}{\partial X} \right| \tag{2.3}$$

the equations now becomes

$$\frac{\partial}{\partial t}(Jw) + \frac{\partial F_j}{\partial X_j} = 0 \tag{2.4}$$

where

$$F_j = J \frac{\partial X_j}{\partial x_i} \left(f_i - umesh_i w \right) \tag{2.5}$$

and in conventional notation

$$\begin{aligned} \frac{\partial I}{\partial x} &= \frac{\partial X_1}{\partial x_1} = -\frac{\partial y}{\partial I} \\ \frac{\partial I}{\partial y} &= \frac{\partial X_1}{\partial x_2} = +\frac{\partial x}{\partial I} \\ \frac{\partial J}{\partial x} &= \frac{\partial X_2}{\partial x_1} = +\frac{\partial y}{\partial J} \\ \frac{\partial J}{\partial y} &= \frac{\partial X_2}{\partial x_2} = -\frac{\partial x}{\partial J} \end{aligned} \tag{2.6}$$

in integral form

$$\frac{\partial}{\partial t} \int_{\mathcal{D}} w d\mathcal{V} + \int_{\mathcal{B}} \left(f_i - umesh_i w \right) d\mathcal{S}_i = 0 \tag{2.7}$$

where $d\mathcal{S}_i$ is the component of area projected in the x_i direction. a set of ODE's can be obtained from discretisation of spatial variables

$$\frac{d}{dt} \left(w\mathcal{V} \right) + R\left(w \right) = 0 \tag{2.8}$$

discretize this equation in time implicitly,

$$\frac{3}{2\Delta t} w_{i,j}^{n+1} \mathcal{V}_{i,j}^{n+1} - \frac{4}{2\Delta t} w_{i,j}^{n} \mathcal{V}_{i,j}^{n} + \frac{1}{2\Delta t} w_{i,j}^{n-1} \mathcal{V}_{i,j}^{n-1} + R\left(w_{i,j}^{n+1}\right) = 0 \tag{2.9}$$

we then linearize $R\left(w_{i,j}^{n+1}\right)$

$$R(w^{n+1}) = R(w^n) + \frac{\partial R(w^n)}{\partial w} \Delta w^n + \mathcal{O}\, ||\Delta w||^2 \tag{2.10}$$

but we know that

$$\frac{\partial R(w^n)}{\partial w} = \frac{\partial}{\partial w}\left(\frac{\partial F_i}{\partial X_i}\right) = \frac{\partial}{\partial X_i}\frac{\partial F_i}{\partial w} = \frac{\partial}{\partial X_i} A_i \tag{2.11}$$

combining equations we get

$$\begin{aligned} &\frac{3}{2\Delta t} w_{i,j}^{n+1} \mathcal{V}_{i,j}^{n+1} - \frac{4}{2\Delta t} w_{i,j}^{n} \mathcal{V}_{i,j}^{n} + \frac{1}{2\Delta t} w_{i,j}^{n-1} \mathcal{V}_{i,j}^{n-1} \\ &+ \left(\frac{\partial}{\partial X_1} A_{i,j}^n + \frac{\partial}{\partial X_2} B_{i,j}^n\right) \Delta w^n + R\left(w_{i,j}^n\right) = 0 \end{aligned} \tag{2.12}$$

setting

$$\mathcal{V}_{i,j} = \mathcal{V}_{i,j}^{n+1} = \mathcal{V}_{i,j}^{n} = \mathcal{V}_{i,j}^{n-1} \tag{2.13}$$

and multiply everything by $\frac{2\Delta t}{3\mathcal{V}}$ gives us

$$\begin{aligned} w_{i,j}^{n+1} - \frac{4}{3} w_{i,j}^{n} + \frac{1}{3} w_{i,j}^{n-1} \\ + \frac{2\Delta t}{3}\left(D_I A + D_J B\right) \Delta w^n + \frac{2\Delta t}{3} R\left(w_{i,j}^n\right) = \mathcal{O}\, ||\Delta w||^2 \end{aligned} \tag{2.14}$$

substitute $w_{i,j}^{n+1} - w_{i,j}^{n}$ by $\Delta w_{i,j}^n$ and rearrange equations,

$$\left\{I + \frac{2\Delta t}{3\mathcal{V}}\left(D_I A + D_J B\right)\right\} \Delta w_{i,j}^n = \frac{1}{3}\Delta w_{i,j}^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R\left(w_{i,j}^n\right) + \mathcal{O}\, ||\Delta w||^2 \tag{2.15}$$

factorized form is

$$\left(I + \frac{2\Delta t}{3\mathcal{V}} D_I A\right)\left(I + \frac{2\Delta t}{3\mathcal{V}} D_J B\right) \Delta w_{i,j}^n = \frac{1}{3}\Delta w_{i,j}^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R(w_{i,j}^n) + \mathcal{O}\, ||\Delta w||^2 \tag{2.16}$$

we then solve the factorized form in 2 steps, first find $\overline{\Delta w}$

$$\left(I + \frac{2\Delta t}{3\mathcal{V}} D_I A\right) \overline{\Delta w}_{i,j}^n = \frac{1}{3}\Delta w_{i,j}^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R(w_{i,j}^n) \tag{2.17}$$

then solve for $\Delta w_{i,j}^n$

$$\left(I + \frac{2\Delta t}{3\mathcal{V}} D_J B\right) \Delta w_{i,j}^n = \overline{\Delta w}_{i,j}^n \tag{2.18}$$

2.2 Matrix Representation

Overall, for the ADI type methods, where the tridiagonal Jacobian matrices looks like below,

$$\begin{bmatrix} D & C & 0 & 0 & 0 & 0 \\ A & D & C & 0 & 0 & 0 \\ 0 & A & D & C & 0 & 0 \\ 0 & 0 & A & D & C & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \\ \Delta w_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots \\ \text{R.H.S.} \\ \cdots \end{bmatrix} \tag{2.19}$$

the diffusion of the difference of the fluxes at $j+1$ and j written in first order expansion of Taylor series,

$$\begin{aligned} &\overbrace{\frac{\partial\left(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\right)}{\partial w_{j-1}}}^{A} \Delta w_{j-1} + \overbrace{\frac{\partial\left(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\right)}{\partial w_{j}}}^{D} \Delta w_{j} + \overbrace{\frac{\partial\left(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\right)}{\partial w_{j+1}}}^{C} \Delta w_{j+1} \\ &= A^-_{j+1}\Delta w_{j+1} - A^-_j \Delta w_j + A^+_j \Delta w_j - A^+_{j-1}\Delta w_{j-1} \\ &= \left(D^+_x A^- + D^-_x A^+\right)\Delta w \\ &= \underbrace{\left(-A^+_{j-1}\right)}_{A}\Delta w_{j-1} + \underbrace{\left(A^+_j - A^-_j\right)}_{D}\Delta w_j + \underbrace{\left(A^-_{j+1}\right)}_{C}\Delta w_{j+1} \end{aligned} \tag{2.20}$$

this results in

$$\begin{bmatrix} -A^-_2 + A^+_2 & A^-_3 & 0 & 0 & 0 & 0 \\ -A^+_2 & -A^-_3 + A^+_3 & A^-_4 & 0 & 0 & 0 \\ 0 & -A^+_3 & -A^-_4 + A^+_4 & A^-_5 & 0 & 0 \\ 0 & 0 & -A^+_4 & -A^-_5 + A^+_5 & A^-_6 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \\ \Delta w_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots \\ \text{R.H.S.} \\ \cdots \end{bmatrix} \tag{2.21}$$

Therefore, flux splitting is equivalent to added diffusion. Notice that each Jacobian is needed twice (i.e., in two rows of the matrix), therefore, it is better to calculate all the Jacobians and store them in memory than to calculate them on the fly. The equations written in matrix form, first solve for $\overline{\Delta w}$ by solving for each j a set of equations

$$\left[[I] + \frac{2\Delta t}{3\mathcal{V}}\left[D_x A\right]\right] \begin{bmatrix} \overline{\Delta w}^n_{i=2} \\ \overline{\Delta w}^n_{i=3} \\ \overline{\Delta w}^n_{i=4} \\ \overline{\Delta w}^n_{i=5} \\ \vdots \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \Delta w^{n-1}_{i=2} \\ \Delta w^{n-1}_{i=3} \\ \Delta w^{n-1}_{i=4} \\ \Delta w^{n-1}_{i=5} \\ \vdots \end{bmatrix} - \frac{2\Delta t}{3\mathcal{V}} \begin{bmatrix} R\left(w^n_{i=2}\right) \\ R\left(w^n_{i=3}\right) \\ R\left(w^n_{i=4}\right) \\ R\left(w^n_{i=5}\right) \\ \vdots \end{bmatrix} \tag{2.22}$$

where

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.23}$$

and

$$[D_x A] = \begin{bmatrix} -A_2^- + A_2^+ & A_3^- & 0 & 0 & 0 & 0 \\ -A_2^+ & -A_3^- + A_3^+ & A_4^- & 0 & 0 & 0 \\ 0 & -A_3^+ & -A_4^- + A_4^+ & A_5^- & 0 & 0 \\ 0 & 0 & -A_4^+ & -A_5^- + A_5^+ & A_6^- & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.24}$$

of course, special boundary conditions have to be applied at $i = 2$ and $i = IL$. Then solve for Δw^n by sweeping the other direction, i.e. for each i, solve the following

$$\left[[I] + \frac{2\Delta t}{3\mathcal{V}} [D_y B] \right] \begin{bmatrix} \Delta w^n_{j=2} \\ \Delta w^n_{j=3} \\ \Delta w^n_{j=4} \\ \Delta w^n_{j=5} \\ \vdots \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \overline{\Delta w}^n_{j=2} \\ \overline{\Delta w}^n_{j=3} \\ \overline{\Delta w}^n_{j=4} \\ \overline{\Delta w}^n_{j=5} \\ \vdots \end{bmatrix} \tag{2.25}$$

where

$$[D_y B] = \begin{bmatrix} -B_2^- + B_2^+ & B_3^- & 0 & 0 & 0 & 0 \\ -B_2^+ & -B_3^- + B_3^+ & B_4^- & 0 & 0 & 0 \\ 0 & -B_3^+ & -B_4^- + B_4^+ & B_5^- & 0 & 0 \\ 0 & 0 & -B_4^+ & -B_5^- + B_5^+ & B_6^- & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.26}$$

again, we may treat the boundary conditions at $j = 2$ and $j = JL$ implicitly.

2.3 Flux Splitting and Diffusion Equivalence

The flux splitting of A^+, A^-, B^+ and B^- are done by upwinding the eigenvalues of the Jacobian matrices, i.e.

$$A^{\pm} = \frac{1}{2} (A \pm 2\epsilon_i I) \tag{2.27}$$

$$B^{\pm} = \frac{1}{2} (B \pm 2\epsilon_j I) \tag{2.28}$$

where ϵ_j is the diffusion coefficient from the artificial dissipation of the JST scheme's first order diffusion term, then

$$A^+ + A^- = A \tag{2.29}$$

$$B^+ + B^- = B \tag{2.30}$$

Here, the Jacobians can be obtained from simple transformations $A = \hat{M}\hat{A}\hat{M}^{-1}$ as derived by Jameson. The left hand side can also be derived from taking the partial derivative of the fluxes with respect to the corresponding state variable.

$$[D_yB] = \begin{bmatrix} \frac{\partial\left(h_{2\frac{1}{2}} - h_{1\frac{1}{2}}\right)}{\partial w_2} & \frac{\partial\left(h_{2\frac{1}{2}} - h_{1\frac{1}{2}}\right)}{\partial w_3} & 0 & 0 & 0 & 0 \\ \frac{\partial\left(h_{3\frac{1}{2}} - h_{2\frac{1}{2}}\right)}{\partial w_2} & \frac{\partial\left(h_{3\frac{1}{2}} - h_{2\frac{1}{2}}\right)}{\partial w_3} & \frac{\partial\left(h_{3\frac{1}{2}} - h_{2\frac{1}{2}}\right)}{\partial w_4} & 0 & 0 & 0 \\ 0 & \frac{\partial\left(h_{4\frac{1}{2}} - h_{3\frac{1}{2}}\right)}{\partial w_3} & \frac{\partial\left(h_{4\frac{1}{2}} - h_{3\frac{1}{2}}\right)}{\partial w_4} & \frac{\partial\left(h_{4\frac{1}{2}} - h_{3\frac{1}{2}}\right)}{\partial w_5} & 0 & 0 \\ 0 & 0 & \frac{\partial\left(h_{5\frac{1}{2}} - h_{4\frac{1}{2}}\right)}{\partial w_4} & \frac{\partial\left(h_{5\frac{1}{2}} - h_{4\frac{1}{2}}\right)}{\partial w_5} & \frac{\partial\left(h_{5\frac{1}{2}} - h_{4\frac{1}{2}}\right)}{\partial w_6} & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.31}$$

With first order diffusion $h_{j+\frac{1}{2}} = f_{j+\frac{1}{2}} - d_{j+\frac{1}{2}}$, the Flux Split and the above formulations are equivalent.

3 Dual-Time Stepping

Discretize

$$\frac{\partial w}{\partial t} + R(w) = 0, \quad R(w) = \frac{\partial}{\partial x} f(w) + \frac{\partial}{\partial y} g(w) \tag{3.1}$$

by the fully implicit scheme

$$\frac{3}{2\Delta t} w^{n+1} - \frac{2}{\Delta t} w^n + \frac{1}{2\Delta t} w^{n-1} + R(w^{n+1}) = 0. \tag{3.2}$$

This is second order accurate and A-stable (the third order accurate scheme is stiffly stable). Using an inner iteration, solve

$$\frac{\partial w}{\partial \tau} + \frac{3}{2\Delta t} w + R(w) - \frac{2}{\Delta t} w^n + \frac{1}{2\Delta t} w^{n-1} = 0 \tag{3.3}$$

to pseudo-time, τ, steady-state with

1. explicit multistage scheme
2. variable local $\Delta\tau$
3. implicit residual averaging
4. multigrid

The main snag with this scheme is that no error estimate for time accuracy can be found unless the inner iteration is fully converged.

4 Proposed Hybrid Scheme

The 2D Euler or Navier-Stokes equations in conservation form

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} + \frac{\partial g(w)}{\partial y} = 0,$$

can be discretized using a nonlinear implicit scheme as

$$\frac{3}{2\Delta t} w^{n+1} - \frac{2}{\Delta t} w^{n} + \frac{1}{2\Delta t} w^{n-1} + R(w^{n+1}) = 0, \tag{4.1}$$

where the discrete residual

$$R(w) = D_x f(w) + D_y g(w).$$

This can be linearized by setting

$$R(w^{n+1}) = R(w^{n}) + \frac{\partial R(w^n)}{\partial w} \Delta w^n + \mathcal{O} \left|\left|\Delta w\right|\right|^2$$

or

$$R(w^{n+1}) = R(w^{n}) + (D_x A + D_y B) \Delta w + \mathcal{O} \left|\left|\Delta w\right|\right|^2, \tag{4.2}$$

where

$$\begin{aligned} \Delta w^n &= w^{n+1} - w^n \\ A &= \frac{\partial f(w)}{\partial w} \\ B &= \frac{\partial g(w)}{\partial w}. \end{aligned}$$

Substitute the fourth term $R(w^{n+1})$ in (4.1) by (4.2), and we have

$$\begin{aligned} \frac{3}{2\Delta t}\Delta w^n - \frac{1}{2\Delta t}\Delta w^{n-1} &+ R(w^n) + (D_x A + D_y B)\Delta w^n \\ &= \frac{\partial w}{\partial t} + R(w^n) + \mathcal{O}(\Delta t^2). \end{aligned} \tag{4.3}$$

We can rearrange the terms in (4.3) to get

$$\begin{aligned} &\left\{ I + \frac{2\Delta t}{3}(D_x A + D_y B) \right\} \Delta w^n + \frac{2\Delta t}{3} R(w^n) - \frac{1}{3}\Delta w^{n-1} \\ &\qquad = \frac{2\Delta t}{3}\left(\frac{\partial w}{\partial t} + R(w^n) + O(\Delta t^2) \right). \end{aligned} \tag{4.4}$$

The proposed hybrid scheme will take an initial ADI step

$$\left(I + \frac{2\Delta t}{3} D_x A\right)\left(I + \frac{2\Delta t}{3} D_y B\right) \Delta w^{(1)} + \frac{2\Delta t}{3} R(w^n) - \frac{1}{3}\Delta w^{n-1} = 0, \tag{4.5}$$

and then iterate with a multistage time stepping scheme augmented by multigrid to drive the solution in the steady state limit towards the linearized equation (4.3).

$$\Delta w^{(k)} - \Delta w^{(k-1)} + \beta \left\{ \left(I + \frac{2\Delta t}{3}(D_x A + D_y B) \right) \Delta w^{(k-1)} + \frac{2\Delta t}{3} R(w^n) - \frac{1}{3}\Delta w^{n-1} \right\} = 0 \quad (4.6)$$

the iteration can be expanded as follows.

$$\Delta w^{(2)} - \Delta w^{(1)} + \beta_1 \left\{ \left(I + \frac{2\Delta t}{3}(D_x A + D_y B) \right) \Delta w^{(1)} + \frac{2\Delta t}{3} R(w^n) - \frac{1}{3}\Delta w^{n-1} \right\} = 0 \quad (4.7)$$

$$\Delta w^{(3)} - \Delta w^{(2)} + \beta_2 \left\{ \left(I + \frac{2\Delta t}{3}(D_x A + D_y B) \right) \Delta w^{(2)} + \frac{2\Delta t}{3} R(w^n) - \frac{1}{3}\Delta w^{n-1} \right\} = 0 \quad (4.8)$$

$$\cdots$$

$$\cdots$$

The initial ADI step is already formally $\mathcal{O}(\Delta t^2)$, and subtracting the product of β_1 and (4.5) from (4.7) we get

$$\Delta w^{(2)} - \Delta w^{(1)} = \beta_1 \frac{4\Delta t^2}{9} D_x A D_y B \Delta w^{(1)} = \mathcal{O}(\Delta t^2), \quad (4.9)$$

and subsequently any $\Delta w^{(k)} - \Delta w^{(k-1)}$ is also $O(\Delta t^2)$.

The advantages of this scheme are that

1. We should retain formal second order accuracy with any number of iterations, and it should not be necessary to iterate to convergence within each implicit time step, in contrast to existing dual-time stepping schemes which are only second order accurate if the inner iterations are fully converged.

2. The additional iterations with multigrid should provide information exchange between processors which is needed to stabilize the ADI scheme run separately in each processor.

5 Multigrid for ADI

We can apply multigrid to ADI if necessary. On the finest mesh,

$$\overbrace{\left(I+\frac{2\Delta t}{3\mathcal{V}}D_xA\right)\left(I+\frac{2\Delta t}{3\mathcal{V}}D_yB\right)\Delta w_{i,j}^n}^{\mathcal{L}(w^n)}+\overbrace{\frac{2\Delta t}{3\mathcal{V}}R\left(w_{i,j}^n\right)-\frac{1}{3}\Delta w_{i,j}^{n-1}}^{\Delta tR^*(w^n)}=0 \quad (5.1)$$

on the coarser meshes, we'll solve the modified equation,

$$\overbrace{\left(I+\frac{2\Delta t}{3\mathcal{V}}D_xA\right)_{2h}\left(I+\frac{2\Delta t}{3\mathcal{V}}D_yB\right)_{2h}\Delta w_{2h}^n}^{\mathcal{L}_{2h}(w_{2h}^n)}+\Delta t\left(R_{2h}^*\left(w_{2h}^{(0)}\right)+\overbrace{\left[\sum R_h^*\left(w_h^n\right)-R_{2h}^*\left(w_{2h}^{(0)}\right)\right]}^{WR}\right)=0 (5.2)$$

(the multigrid calculation is similar to the explicit method.)

In the V-cycle, a brief description of what happens

1. update in EULER $W^n \to W^{n+1}$
2. calculate residuals DW in EULER
3. restrict both W^{n+1} and DW to coarser mesh
4. in coarser mesh, update in EULER $W^n \to W^{n+1}$ with right hand side modified as in (5.2). The residual calculation is needed to find the RHS.
5. calculate residual of updated value in EULER.
6. continue to descent to coarser mesh.
7. when ascending back to finer mesh, prolongate the TOTAL CHANGE in W in the coarser mesh, and add it to the updated value in the finer mesh.

6 Stability Analysis of ADI

Suppose we start with the discretized equation

$$\left(I+\mu\frac{2\Delta t}{3\mathcal{V}}D_xA\right)\left(I+\mu\frac{2\Delta t}{3\mathcal{V}}D_yB\right)\Delta w_{i,j}^n=\frac{1}{3}\Delta w_{i,j}^{n-1}-\frac{2\Delta t}{3\mathcal{V}}R(w_{i,j}^n)+\mathcal{O}\left|\left|\Delta w\right|\right|^2 \tag{6.1}$$

where the constant μ is added for the sake of changing the stability of the scheme. It comes from the equation

$$\frac{3}{2\Delta t}w^{n+1}-\frac{2}{\Delta t}w^n+\frac{1}{2\Delta t}w^{n-1}+\mu R(w^{n+1})+(1-\mu)\,R(w^n)=0 \tag{6.2}$$

substituting the Fourier modes $w_{i,j}^n=g^ne^{i\omega_xX_i}e^{i\omega_yY_j}$ for the scalar case, letting $\lambda_x=A\frac{\Delta t}{\Delta x}$ and $\lambda_y=B\frac{\Delta t}{\Delta y}$, and let $\xi_x=\omega_x\Delta x$ and $\xi_y=\omega_y\Delta y$. and factoring out $g^ne^{i\omega_xX_i}e^{i\omega_yY_j}$

$$(g-1)\left[\frac{3}{2}-\frac{1}{2g}+i\mu\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}\right]+i\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}=0 \tag{6.3}$$

then if we attempt to solve for g by rewriting in quadratic form

$$\left[\frac{3}{2}+i\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}\right]g^2+\left[i\left(1-\mu\right)\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}-2\right]g+\frac{1}{2}=0 \tag{6.4}$$

the roots for this equation are

$$\frac{\left(\left(\mu-1\right)-i\left(2\mu-4\right)\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}\right)\pm\sqrt{\left(\left(\mu-1\right)-i\left(2\mu-4\right)\right)\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}-1}}{3+2i\mu\left\{\lambda_x sin\left(\xi_x\right)+\lambda_y sin\left(\xi_y\right)\right\}} \tag{6.5}$$

7 Boundary Treatment for Inviscid Euler Solid Body

From the flow tangency condition for the flux through solid wall,

$$S_x \frac{(u_1 + u_2)}{2} + S_y \frac{(v_1 + v_2)}{2} = 0 \tag{7.1}$$

therefore the only remaining terms of the boundary flux is

$$g_{1\frac{1}{2}} = \begin{bmatrix} 0 \\ -Y_\xi (P_2 + P_1) \\ X_\xi (P_2 + P_1) \\ q_s^+ P_2 + q_s^- P_1 \end{bmatrix} \tag{7.2}$$

where for J-direction flux

$$S_x = -Y_\xi \tag{7.3}$$
$$S_y = X_\xi \tag{7.4}$$
$$q_s^+ = S_x u_2 + S_y v_2 \tag{7.5}$$
$$q_s^- = S_x u_1 + S_y v_1 \tag{7.6}$$
$$P_1 = P_2 = (\gamma - 1) \left((\rho E)_2 - \frac{(\rho u)_2^2 + (\rho v)_2^2}{2\rho_2} \right) \tag{7.7}$$

where the subscripts 1 denotes the cell immediately outside the wall, 2 denotes the cell immediately inside the wall, and the metrics are calculated at the boundary surface. Therefore, the flux Jacobians are derived by taking derivatives of the flux with respect to the state variables of respective cells, i.e.

$$B_1^+ = \frac{\partial g_{1\frac{1}{2}}}{\partial W_1} \tag{7.8}$$
$$B_2^- = \frac{\partial g_{1\frac{1}{2}}}{\partial W_2} \tag{7.9}$$

and it is easy to rewrite B_1^+ in terms of state variables W_2.

8 Old Boundary Treatment for Inviscid Euler Solid Body

The implicit inviscid wall BC needs special attention, from the constitutive equations at the boundary,

$$\delta\rho_1 = \delta\rho_2 \tag{8.1}$$
$$\delta\rho_1 U_1 = \delta\rho_1 U_2 \tag{8.2}$$
$$\delta\rho_1 V_1 = \delta\rho_1 V_2 \tag{8.3}$$
$$\delta\rho_1 E_1 = \delta\rho_1 E_2 \tag{8.4}$$

where U_1, V_1 and U_2, V_2 are the contravariant velocities in the flow and in the solid body respectively.

$$\rho_1 U_1 = X_{x_{\frac{3}{2}}} \rho_1 u_1 + X_{y_{\frac{3}{2}}} \rho_1 v_1 \tag{8.5}$$
$$\rho_2 U_2 = X_{x_{\frac{3}{2}}} \rho_2 u_2 + X_{y_{\frac{3}{2}}} \rho_2 v_2 \tag{8.6}$$
$$\rho_1 V_1 = Y_{x_{\frac{3}{2}}} \rho_1 u_1 + Y_{y_{\frac{3}{2}}} \rho_1 v_1 \tag{8.7}$$
$$\rho_2 V_2 = Y_{x_{\frac{3}{2}}} \rho_2 u_2 + Y_{y_{\frac{3}{2}}} \rho_2 v_2 \tag{8.8}$$

where the metrics are calculated at the surface of the boundary, and the densities and absolute velocities taken from the cell centers. Taking the difference,

$$\delta\rho_1 U_1 = X_{x_{\frac{3}{2}}} \delta\left(\rho_1 u_1\right) + X_{y_{\frac{3}{2}}} \delta\left(\rho_1 v_1\right) \tag{8.9}$$
$$\delta\rho_2 U_2 = X_{x_{\frac{3}{2}}} \delta\left(\rho_2 u_2\right) + X_{y_{\frac{3}{2}}} \delta\left(\rho_2 v_2\right) \tag{8.10}$$
$$\delta\rho_1 V_1 = Y_{x_{\frac{3}{2}}} \delta\left(\rho_1 u_1\right) + Y_{y_{\frac{3}{2}}} \delta\left(\rho_1 v_1\right) \tag{8.11}$$
$$\delta\rho_2 V_2 = Y_{x_{\frac{3}{2}}} \delta\left(\rho_2 u_2\right) + Y_{y_{\frac{3}{2}}} \delta\left(\rho_2 v_2\right) \tag{8.12}$$

relating the above equations,

$$X_{x_{\frac{3}{2}}} \delta\left(\rho_1 u_1\right) + X_{y_{\frac{3}{2}}} \delta\left(\rho_1 v_1\right) = X_{x_{\frac{3}{2}}} \delta\left(\rho_2 u_2\right) + X_{y_{\frac{3}{2}}} \delta\left(\rho_2 v_2\right) \tag{8.13}$$
$$Y_{x_{\frac{3}{2}}} \delta\left(\rho_1 u_1\right) + Y_{y_{\frac{3}{2}}} \delta\left(\rho_1 v_1\right) + Y_{x_{\frac{3}{2}}} \delta\left(\rho_2 u_2\right) + Y_{y_{\frac{3}{2}}} \delta\left(\rho_2 v_2\right) = 0 \tag{8.14}$$

rearrange the terms,

$$X_{x_{\frac{3}{2}}} \left(\delta\left(\rho_1 u_1\right) - \delta\left(\rho_2 u_2\right)\right) + X_{y_{\frac{3}{2}}} \left(\delta\left(\rho_1 v_1\right) - \delta\left(\rho_2 v_2\right)\right) = 0 \tag{8.15}$$
$$Y_{x_{\frac{3}{2}}} \left(\delta\left(\rho_1 u_1\right) + \delta\left(\rho_2 u_2\right)\right) + Y_{y_{\frac{3}{2}}} \left(\delta\left(\rho_1 v_1\right) + \delta\left(\rho_2 v_2\right)\right) = 0 \tag{8.16}$$

in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{x_{\frac{3}{2}}} & X_{x_{\frac{3}{2}}} & 0 \\ 0 & Y_{x_{\frac{3}{2}}} & Y_{x_{\frac{3}{2}}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\left(\rho_1\right) \\ \delta\left(\rho_1 u_1\right) \\ \delta\left(\rho_1 v_1\right) \\ \delta\left(\rho_1 E_1\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{x_{\frac{3}{2}}} & X_{x_{\frac{3}{2}}} & 0 \\ 0 & -Y_{x_{\frac{3}{2}}} & -Y_{x_{\frac{3}{2}}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\left(\rho_2\right) \\ \delta\left(\rho_2 u_2\right) \\ \delta\left(\rho_2 v_2\right) \\ \delta\left(\rho_2 E_2\right) \end{bmatrix} \tag{8.17}$$

solving for $\begin{bmatrix} \delta(\rho_2) \\ \delta(\rho_2 u_2) \\ \delta(\rho_2 v_2) \\ \delta(\rho_2 E_2) \end{bmatrix}$ in terms of $\begin{bmatrix} \delta(\rho_1) \\ \delta(\rho_1 u_1) \\ \delta(\rho_1 v_1) \\ \delta(\rho_1 E_1) \end{bmatrix}$ we have

$$\begin{bmatrix} \delta(\rho_2) \\ \delta(\rho_2 u_2) \\ \delta(\rho_2 v_2) \\ \delta(\rho_2 E_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{x_{\frac{3}{2}}} & X_{x_{\frac{3}{2}}} & 0 \\ 0 & -Y_{x_{\frac{3}{2}}} & -Y_{x_{\frac{3}{2}}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & X_{x_{\frac{3}{2}}} & X_{x_{\frac{3}{2}}} & 0 \\ 0 & Y_{x_{\frac{3}{2}}} & Y_{x_{\frac{3}{2}}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta(\rho_1) \\ \delta(\rho_1 u_1) \\ \delta(\rho_1 v_1) \\ \delta(\rho_1 E_1) \end{bmatrix} \tag{8.18}$$

and inverting the $2X2$ matrix,

$$\begin{bmatrix} \delta(\rho_2) \\ \delta(\rho_2 u_2) \\ \delta(\rho_2 v_2) \\ \delta(\rho_2 E_2) \end{bmatrix} = \frac{1}{\mathcal{A}} \overbrace{\begin{bmatrix} \mathcal{A} & 0 & 0 & 0 \\ 0 & (Y_{y_{\frac{3}{2}}} X_{x_{\frac{3}{2}}} + X_{y_{\frac{3}{2}}} Y_{x_{\frac{3}{2}}}) & (Y_{y_{\frac{3}{2}}} X_{y_{\frac{3}{2}}} + X_{y_{\frac{3}{2}}} Y_{y_{\frac{3}{2}}}) & 0 \\ 0 & -(Y_{x_{\frac{3}{2}}} X_{x_{\frac{3}{2}}} - X_{x_{\frac{3}{2}}} Y_{x_{\frac{3}{2}}}) & -(Y_{x_{\frac{3}{2}}} X_{y_{\frac{3}{2}}} - X_{x_{\frac{3}{2}}} Y_{y_{\frac{3}{2}}}) & 0 \\ 0 & 0 & 0 & \mathcal{A} \end{bmatrix}}^{\mathcal{H}} \begin{bmatrix} \delta(\rho_1) \\ \delta(\rho_1 u_1) \\ \delta(\rho_1 v_1) \\ \delta(\rho_1 E_1) \end{bmatrix} \tag{8.19}$$

where

$$\mathcal{A} = \frac{1}{X_{x_{\frac{3}{2}}} Y_{y_{\frac{3}{2}}} - X_{y_{\frac{3}{2}}} X_{x_{\frac{3}{2}}}} \tag{8.20}$$

from this, the tridiagonal matrix for the j direction with fixed i can be re-written as (written here for the second sweep, but can very well be the first sweeping direction, just change the unknowns and the R.H.S.). Ideally, this should be done as the initial sweep, since the intermediate variables $\overline{\Delta w_{i,j}^n}$ may not share the same type of boundary conditions.

$$\left[[I] + \frac{2\Delta t}{3\mathcal{V}} \begin{bmatrix} -\overbrace{B_2^-}^{note1} + B_2^+ - \overbrace{B_1^+}^{note2} \mathcal{H} & B_3^- & 0 & 0 \\ -B_2^+ & -B_3^- + B_3^+ & B_4^- & 0 \\ 0 & -B_3^+ & -B_4^- + B_4^+ & B_5^- \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \right] \begin{bmatrix} \Delta w_{j=2}^{n-1} \\ \Delta w_{j=3}^{n-1} \\ \Delta w_{j=4}^{n-1} \\ \Delta w_{j=5}^{n-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \overline{\Delta w}_{j=2}^n \\ \overline{\Delta w}_{j=3}^n \\ \overline{\Delta w}_{j=4}^n \\ \overline{\Delta w}_{j=5}^n \\ \vdots \end{bmatrix} \tag{8.21}$$

The above *note*1 and *note*2 comes from

$$\begin{aligned} B_2^- &\leftarrow \frac{\partial g_{1\frac{1}{2}}}{\partial w_2} \\ B_1^+ &\leftarrow \frac{\partial g_{1\frac{1}{2}}}{\partial w_1} \end{aligned} \tag{8.22}$$

further, if $g_{1\frac{1}{2}}$ depends only on w_2, then $B_1^+ = 0$.

9 LU Decomposition of Block Tridiagonal with Periodic Boundaries

For a O-mesh, this is necessary for $I = 1$ and $I = IE$ in J-sweeps. Starting with

$$\begin{bmatrix} d_1 & c_1 & 0 & 0 & 0 & 0 & \dots & 0 & a_1 \\ a_2 & d_2 & c_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_3 & d_3 & c_3 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_4 & d_4 & c_4 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & & a_{n-3} & d_{n-3} & c_{n-3} & 0 & 0 \\ 0 & 0 & \dots & & & a_{n-2} & d_{n-2} & c_{n-2} & 0 \\ 0 & 0 & \dots & & & 0 & a_{n-1} & d_{n-1} & c_{n-1} \\ c_n & 0 & \dots & & & 0 & 0 & a_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \\ b_n \end{bmatrix} \tag{9.1}$$

we perform an LU decomposition of the LHS matrix into its components

$$\overbrace{\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 & 0 \\ K_2 & L_2 & 0 & 0 & 0 & 0 \\ 0 & K_3 & L_3 & 0 & 0 & 0 \\ 0 & 0 & K_4 & L_4 & 0 & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ J_1 & J_2 & \dots & J_{n-2} & K_n & L_n \end{bmatrix}}^{\mathcal{L}} \overbrace{\begin{bmatrix} U_1 & V_1 & 0 & 0 & 0 & W_1 \\ 0 & U_2 & V_2 & 0 & 0 & W_2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & & U_{n-2} & V_{n-2} & W_{n-2} \\ 0 & 0 & \dots & 0 & U_{n-1} & V_{n-1} \\ 0 & 0 & \dots & 0 & 0 & U_n \end{bmatrix}}^{\mathcal{U}} \tag{9.2}$$

From the above decomposition, we can derive the following equations:
From the Upper Diagonal elements,

$$\begin{aligned} L_1 V_1 &= C_1 \\ L_2 V_2 &= C_2 \\ \dots & \\ L_{n-2} V_{n-2} &= C_{n-2} \qquad (9.3a) \\ & \mathit{and}, \\ L_{n-1} V_{n-1} + K_{n-1} W_{n-2} &= C_{n-1} \qquad (9.3b) \\ J_1 U_1 &= C_n \qquad (9.3c) \end{aligned}$$

From the Lower Diagonal elements,

$$L_1 W_1 = a_1 \quad (9.4a)$$

and,

$$K_2 U_1 = a_2$$
$$K_3 U_2 = a_3$$
$$\ldots$$
$$K_{n-1} U_{n-2} = a_{n-1} \quad (9.4b)$$

and,

$$K_n U_{n-1} + J_{n-2} V_{n-2} = a_n \quad (9.4c)$$

From the Diagonal elements,

$$L_1 U_1 = d_1 \quad (9.5a)$$

and,

$$L_2 U_2 + K_2 V_1 = d_2$$
$$L_3 U_3 + K_3 V_2 = d_3$$
$$\ldots$$
$$L_{n-1} U_{n-1} + K_{n-1} V_{n-2} = d_{n-1} \quad (9.5b)$$

and,

$$L_n U_n + K_n V_{n-1} + \sum_{i=1}^{n-2} J_i W_i = d_n \quad (9.5c)$$

and from the Zero elements,

$$K_2 W_1 + L_2 W_2 = 0$$
$$K_3 W_2 + L_3 W_3 = 0$$
$$\ldots$$
$$K_{n-2} W_{n-3} + L_{n-2} W_{n-2} = 0 \quad (9.6a)$$

$$J_2 U_2 + J_1 V_1 = 0$$
$$J_3 U_3 + J_2 V_2 = 0$$
$$\ldots$$
$$J_{n-2} U_{n-2} + J_{n-3} V_{n-3} = 0 \quad (9.6b)$$

Solution of this set of equations is provided as follows:

$$L_1 = I$$
$$L_1 = I$$
$$\ldots$$
$$L_n = I \quad (9.7a)$$

$$V_1 = c_1$$
$$V_2 = c_2$$
$$\ldots$$
$$V_{n-2} = c_{n-2} \quad (9.7b)$$

then

$$
\begin{array}{rcl}
U_1 = d_1 & \longrightarrow & K_2 = a_2 U_1^{-1} \\
& \swarrow & \\
U_2 = d_2 - K_2 V_1 & \longrightarrow & K_3 = a_3 U_2^{-1} \\
& \swarrow & \\
U_3 = d_3 - K_3 V_2 & \longrightarrow & K_4 = a_4 U_3^{-1} \\
& \swarrow & \\
& \cdots & \\
& \cdots & \\
U_{n-2} = d_{n-2} - K_{n-2} V_{n-3} & \longrightarrow & K_{n-1} = a_{n-1} U_{n-2}^{-1} \\
& \swarrow & \\
U_{n-1} = d_{n-1} - K_{n-1} V_{n-2} & &
\end{array} \tag{9.8a}
$$

then

$$
\begin{array}{rcl}
W_1 & = & a_1 \qquad (9.9a) \\
& \textit{and} & \\
W_2 & = & -K_2 W_1 \\
W_3 & = & -K_3 W_2 \\
\cdots & & \\
W_{n-2} & = & -K_{n-2} W_{n-3} \qquad (9.9b)
\end{array}
\qquad
\begin{array}{rcl}
J_1 & = & c_n U_1^{-1} \qquad (9.9c) \\
& \textit{and} & \\
J_2 & = & -J_1 V_1 U_2^{-1} \\
J_3 & = & -J_2 V_2 U_3^{-1} \\
\cdots & & \\
J_{n-2} & = & -J_{n-3} V_{n-3} U_{n-2}^{-1} \qquad (9.9d)
\end{array}
$$

and finally

$$V_{n-1} = c_{n-1} - K_{n-1} W_{n-2} \tag{9.10a}$$

$$K_n = \left(a_n - J_{n-2} V_{n-2}\right) U_{n-1}^{-1} \tag{9.10b}$$

$$U_n = d_n - K_n V_{n-1} - \sum_{i=1}^{n-2} J_i W_i \tag{9.10c}$$

Then for the solution of the equation (9.1):

$$[L]\,\overbrace{[U]\,[x]}^{[y]} = [b] \tag{9.11}$$

solve first

$$[L]\,[y] = [b] \tag{9.12}$$

by forward substitution

$$\begin{aligned}
L_1 y_1 &= b_1 \\
K_2 y_1 + L_2 y_2 &= b_2 \\
K_3 y_2 + L_3 y_3 &= b_3 \\
&\cdots \\
K_{n-1} y_{n-2} + L_{n-1} y_{n-1} &= b_{n-1} \\
&\mathit{and,} \\
\sum_{i=1}^{n-2} J_i y_i + K_n y_{n-1} + L_n y_n &= b_n
\end{aligned}$$

$$\begin{aligned}
y_1 &= L_1^{-1} b_1 \\
y_2 &= L_2^{-1} (b_2 - K_2 y_1) \\
y_3 &= L_3^{-1} (b_3 - K_3 y_2) \\
&\cdots \\
y_{n-1} &= L_{n-1}^{-1} (b_{n-1} - K_{n-1} y_{n-2}) \\
&\mathit{and,} \\
y_n &= L_n^{-1} \left(b_n - K_n y_{n-1} - \sum_{i=1}^{n-2} J_i y_i \right)
\end{aligned}$$

and then solve

$$[U]\,[x] = [y] \tag{9.13}$$

by backward substitution

$$\begin{aligned}
U_n x_n &= y_n \\
U_{n-1} x_{n-1} + V_{n-1} x_n &= y_{n-1} \\
&\mathit{and,} \\
U_{n-2} x_{n-2} + V_{n-2} x_{n-1} + W_{n-2} x_n &= y_{n-2} \\
U_{n-3} x_{n-3} + V_{n-3} x_{n-2} + W_{n-3} x_n &= y_{n-3} \\
&\cdots \\
U_1 x_1 + V_1 x_2 + W_1 x_n &= y_1
\end{aligned}$$

$$\begin{aligned}
x_n &= U_n^{-1} y_n \\
x_{n-1} &= U_{n-1}^{-1} (y_{n-1} - V_{n-1} x_n) \\
&\mathit{and,} \\
x_{n-2} &= U_{n-2}^{-1} (y_{n-2} - V_{n-2} x_{n-1} - W_{n-2} x_n) \\
x_{n-3} &= U_{n-3}^{-1} (y_{n-3} - V_{n-3} x_{n-2} - W_{n-3} x_n) \\
&\cdots \\
x_1 &= U_1^{-1} (y_1 - V_1 x_2 - W_1 x_n)
\end{aligned}$$

10 Diagonally Dominant Alternating Direction Implicit (DDADI) Scheme

Consider the upwind flux split,

$$R = D_x^- f^+ + D_x^+ f^- + D_y^- g^+ + D_y^+ g^- \tag{10.1}$$

applied to the Linearized Equation

$$\left\{ I + \frac{\Delta t}{\Delta x}\left(D_x^+ A^- + D_x^- A^+\right) + \frac{\Delta t}{\Delta y}\left(D_y^+ B^- + D_y^- B^+\right) \right\} \Delta w + \Delta t R = 0 \tag{10.2}$$

Setting

$$\begin{aligned} D_x^+ &= (E_x^+ - E_x^0) \\ D_x^- &= (E_x^0 - E_x^-) \\ D_y^+ &= (E_y^+ - E_y^0) \\ D_y^- &= (E_y^0 - E_y^-) \end{aligned}$$

then

$$\left\{ D + \frac{\Delta t}{\Delta x}\left(E_x^+ A^- - E_x^- A^+\right) + \frac{\Delta t}{\Delta y}\left(E_y^+ B^- - E_y^- B^+\right) \right\} \Delta w + \Delta t R = 0 \tag{10.3}$$

where

$$\begin{aligned} A^\pm &= T\Lambda^\pm T^{-1} \\ A^+ - A^- &= T\,|\Lambda|\,T^{-1} = |A| \\ D &= I + \frac{\Delta t}{\Delta x}(A^+ - A^-) + \frac{\Delta t}{\Delta y}(B^+ - B^-) \\ &= I + \frac{\Delta t}{\Delta x}\,|A| + \frac{\Delta t}{\Delta y}\,|B| \end{aligned}$$

remove a factor of D

$$D\left\{ I + D^{-1}\frac{\Delta t}{\Delta x}\left(E_x^+ A^- - E_x^- A^+\right) + D^{-1}\frac{\Delta t}{\Delta y}\left(E_y^+ B^- - E_y^- B^+\right) \right\} \Delta w + \Delta t R = 0 \tag{10.4}$$

and factorize

$$D\left\{ I + D^{-1}\frac{\Delta t}{\Delta x}\left(E_x^+ A^- - E_x^- A^+\right) \right\} \overbrace{\left\{ I + D^{-1}\frac{\Delta t}{\Delta y}\left(E_y^+ B^- - E_y^- B^+\right) \right\} \Delta w}^{\Delta w^{(1)}} + \Delta t R = 0 \tag{10.5}$$

written altrernatively,

$$\left\{ I + \overbrace{\frac{\Delta t}{\Delta y}\,|B|}^{\text{diagonal term}} + \overbrace{\frac{\Delta t}{\Delta x}\left(D_x^+ A^- + D_x^- A^+\right)}^{\frac{\Delta t}{\Delta x}|A| \text{absorbed in here}} \right\} D^{-1} \left\{ I + \overbrace{\frac{\Delta t}{\Delta x}\,|A|}^{\text{diagonal term}} + \overbrace{\frac{\Delta t}{\Delta y}\left(D_y^+ B^- + D_y^- B^+\right)}^{\frac{\Delta t}{\Delta x}|B| \text{absorbed in here}} \right\} \Delta w + \Delta R = 0 \tag{10.6}$$

We can see the two additional diagonal terms when compared to traditional ADI schemes. Solve (10.5) in 2 steps by

1. First solve

$$\left\{D + \frac{\Delta t}{\Delta x}\left(E_x^+ A^- - E_x^- A^+\right)\right\} \Delta w^{(1)} = -\Delta t R \tag{10.7}$$

(alternatively, a better representation of the scheme is)

$$\left\{I + \frac{\Delta t}{\Delta y}|B| + \frac{\Delta t}{\Delta x}\left(D_x^+ A^- + D_x^- A^+\right)\right\} \Delta w^{(1)} = -\Delta R \tag{10.8}$$

2. then solve

$$\left\{D + \frac{\Delta t}{\Delta y}\left(E_y^+ B^- - E_y^- B^+\right)\right\} \Delta w = D\Delta w^{(1)} \tag{10.9}$$

(alternatively, a better representation of the scheme is)

$$\left\{I + \frac{\Delta t}{\Delta x}|A| + \frac{\Delta t}{\Delta y}\left(D_x^+ B^- + D_x^- B^+\right)\right\} \Delta w = D\Delta w^{(1)} \tag{10.10}$$

To investigate the difference between standard Approximate Factorization and the DDADI approach, we use a different approach to derive the DDADI for the ADI with 3-4-1 Backward Differencing in time, starting with

$$\left\{I + \frac{2\Delta t}{3\mathcal{V}}\left(D_x^+ A^- + D_x^- A^+ + D_y^+ B^- + D_y^- B^+\right)\right\} \Delta w = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R \tag{10.11}$$

and extract the diagonal terms before factoring,

$$\left\{D + \frac{2\Delta t}{3\mathcal{V}}\left(E_x^+ A^- - E_x^- A^+ + E_y^+ B^- - E_y^- B^+\right)\right\} \Delta w = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R \tag{10.12}$$

pull the diagonal terms out of the equation

$$D\left\{I + D^{-1}\frac{2\Delta t}{3\mathcal{V}}\left(E_x^+ A^- - E_x^- A^+\right) + D^{-1}\frac{2\Delta t}{3\mathcal{V}}\left(E_y^+ B^- - E_y^- B^+\right)\right\} \Delta w = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R \tag{10.13}$$

and the factorized form is

$$D\left\{I + D^{-1}\frac{2\Delta t}{3\mathcal{V}}\left(E_x^+ A^- - E_x^- A^+\right)\right\}\left\{I + D^{-1}\frac{2\Delta t}{3\mathcal{V}}\left(E_y^+ B^- - E_y^- B^+\right)\right\} \Delta w = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3\mathcal{V}} R \tag{10.14}$$

finally, multiply out the factored form to see what the extra terms are

$$D\left\{I + D^{-1}\frac{2\Delta t}{3\mathcal{V}}\left(E_x^+ A^- - E_x^- A^+ + E_y^+ B^- - E_y^- B^+\right)\right.$$
$$\left. + \underbrace{D^{-1^2}\frac{4\Delta t^2}{9\mathcal{V}^2}\left(E_x^+ A^- - E_x^- A^+\right)\left(E_y^+ B^- - E_y^- B^+\right)}_{\text{extra error term}}\right\}\Delta w = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3\mathcal{V}}R \tag{10.15}$$