An Implicit-Explicit Scheme for Calculating Complex Unsteady Flows

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Industrial Affiliates Meeting 2002
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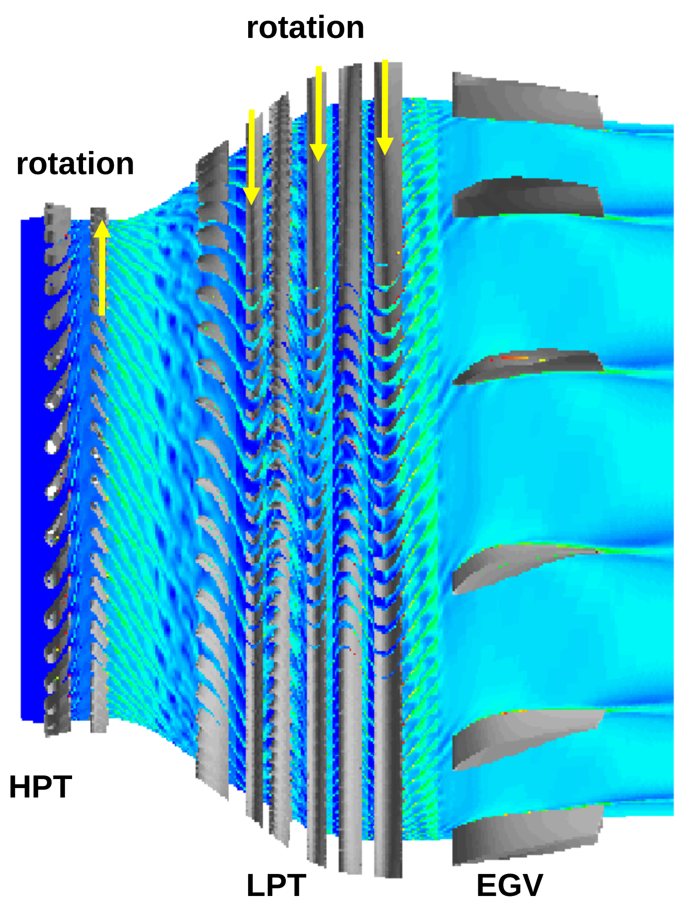
Motivation

- The current calculations of complex unsteady flows are prohibitively expensive for use in real engineering applications to turbomachinery design
- Example:
  - In the Stanford ASCI project, we are calculating the unsteady flow through a complete turbine with 9 blade rows using a mesh with 93.8 million cells and 2192 blocks. Using an implicit scheme, approximately 5700 time steps (each with 30 inner iterations) required to reach a stationary periodic state. The total estimated computer time is 1.04 million CPU hours. Using 384 processors, the calculation requires approximately 4 months.

Unsteady Flow Simulation of High- and Low-Pressure Turbine

Entropy



Contours Scaled Independently for Each Blade Row

Outline

- Fully Implicit Backward Difference Formula (BDF)
- Dual-Time Stepping Scheme (DTSS)
- Linearized Scheme
- ADI-BDF Scheme
- Hybrid Scheme
- Results:
  - 2D Pitching Airfoil Test Case
- Conclusions
- Future work

Fully Implicit Backward Difference Formula (BDF)

Discretize

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} + \frac{\partial g(w)}{\partial y} = 0$$

as the BDF

$$\frac{3}{2\Delta t} w^{n+1} - \frac{4}{2\Delta t} w^{n} + \frac{1}{2\Delta t} w^{n-1} + R(w^{n+1}) = 0$$

where the R(w) is the discrete residual

$$R(w) = D_x f(w) + D_y g(w)$$

Fully Implicit Backward Difference Formula (BDF) contd.

- Advantages:
  - The scheme is second order accurate in time
  - It is A-stable. (i.e. unconditionally stable if the physical equations are stable.)
- Disadvantages:
  - Coupled nonlinear equations have to be solved at each time step by some approximate method.

Fully Implicit Dual Time Stepping Scheme (DTSS)

- Solve the full nonlinear BDF by inner iterations which advance in pseudo time τ

$$\frac{\partial w}{\partial \tau} + \left[\frac{\left(3w^{n+1} - 4w^{n} + w^{n-1}\right)}{2\Delta t} + R(w^{n+1}) \right] = 0$$

alternatively, written in delta form, $\Delta w^n = w^{n+1} - w^n$

$$\frac{\partial w}{\partial \tau} + \left[\frac{3}{2\Delta t}\Delta w^n - \frac{1}{2\Delta t}\Delta w^{n-1} + R(w^{n+1}) \right] = 0$$

on convergence, $\frac{\partial w}{\partial \tau} = 0$,original BDF is recovered.

- We solve DTSS using the fastest known methods
  - Explicit multistage scheme
  - Variable local $\Delta\tau$
  - Implicit residual averaging
  - Multigrid

Fully Implicit Dual Time Stepping Scheme Contd.

- Advantage:
  - If the inner iterations converge fast enough, we solve the fully nonlinear BDF, giving an efficient A-stable scheme which allows very large Δt.
- Disadvantage:
  - No way of assessing accuracy unless the inner iterations are fully converged.
  - If a large number of inner iterations are required the scheme becomes expensive.

Linearized Scheme

Approximate the flux vectors as

$$f(w^{n+1}) = f(w^n) + A\Delta w^n + O||\Delta w||^2$$

$$g(w^{n+1}) = g(w^n) + B\Delta w^n + O||\Delta w||^2$$

where $A = \frac{\partial f(w)}{\partial w}$ and $B = \frac{\partial g(w)}{\partial w}$ are the Jacobians

Hence we obtain the linearized scheme

$$\left[I + \frac{2\Delta t}{3}(D_x A + D_y B)\right]\Delta w^n = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3}R(w^n)$$

Linearized Scheme

- Advantage:
  - Since $||\Delta w|| = O(\Delta t)$ the scheme is still second order accurate.
- Disadvantage:
  - The cost of inversion is still too great.

Alternating Direction Implicit (ADI) Scheme with the Backward Difference Formula (BDF)

Replace the left hand side of the linearized BDF by an approximate factorization, giving the modified ADI scheme

$$\left[I + \frac{2\Delta t}{3} \left(D_x A + D_y B \right) \right] \Delta w^n = \frac{1}{3} \Delta w^{n-1} - \frac{2\Delta t}{3} R(w^n)$$

$$\left(I + \frac{2\Delta \mathrm{t}}{3} D_x A \right) \left(I + \frac{2\Delta \mathrm{t}}{3} D_y B \right) \Delta w^n = \frac{1}{3} \Delta w^{n-1} - \frac{2\Delta t}{3} R(w^n)$$

ADI-BDF Scheme Contd.

- Advantages:
  - Nominally second order accurate in time with 3 sources of error:
    - The discretization error of the BDF
    - The linearization error
    - The factorization error
  - Can be solved at low computational cost in two steps.
- Disadvantages:
  - The factorization error dominates at large CFL numbers
  - The scheme is not amenable to parallel processing: it may lose its stability if applied separately in each of a large numbers of blocks

Hybrid Scheme

The proposed hybrid scheme will take an initial ADI step in real time Δt:

$$\left(I + \frac{2\Delta t}{3} D_x A\right)\left(I + \frac{2\Delta t}{3} D_y B\right)\Delta w^{(1)} = \frac{1}{3}\Delta w^{n-1} - \frac{2\Delta t}{3} R(w^n)$$

yielding a nominal second order accuracy without iterations.

Then follow it with the iterative multistage dual-time stepping scheme augmented by multigrid to drive the solution in the steady state limit towards the fully nonlinear BDF (k=2,3,...):

$$\Delta w^{(k)} - \Delta w^{(k-1)} + \beta_k \left[\frac{3}{2\Delta t}\left(\Delta w^{(k-1)} - \frac{1}{3}\Delta w^{n-1}\right) + R(w^{(k-1)}) \right] = 0$$

where β_k are the constants for the dual-time stepping scheme.

Accuracy of the Hybrid Scheme

The initial ADI step is already formally $O(\Delta t^2)$, and it follows that with the difference between the implicit and explicit steps can be written as follows:

$$\Delta w^{(2)} - \Delta w^{(1)} = \beta_k \left[O\|\Delta w\|^2 + \frac{2\Delta t}{3}(D_x A + D_y B)\Delta w^{(1)} \right]$$

and subsequently, any $\Delta w^{(k)} - \Delta w^{(k-1)}$ is also $O(\Delta t^2)$.

- Advantages:
  - We should retain formal second order accuracy with any number of iterations. Not necessary to iterate to convergence within each implicit time step; in contrast to existing dual-time stepping schemes which are only second order accurate if the inner iterations are fully converged.

Validation

The case selected is the NACA 64A010 airfoil in a pitching oscillation representative of wing flutter:

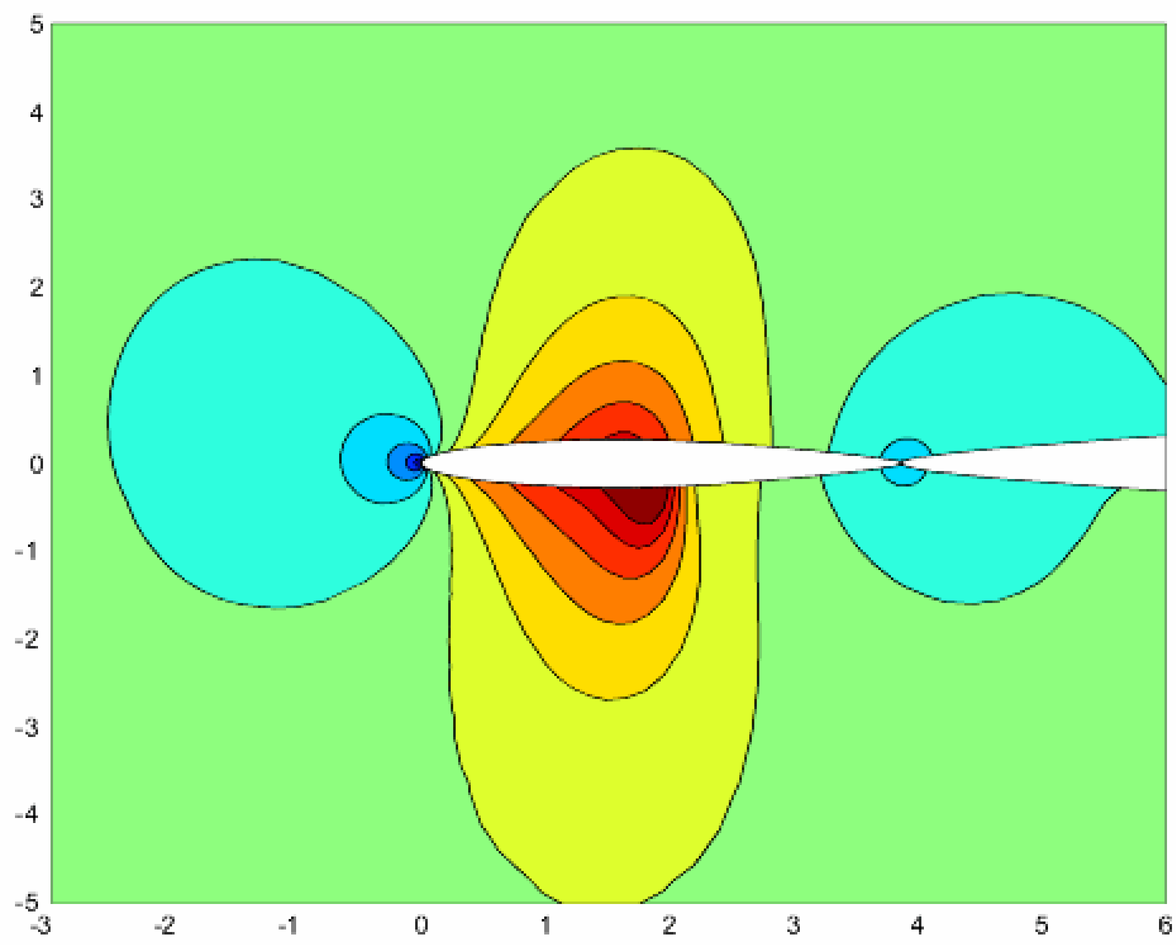
- Mach number: 0.796
- Reduced Frequency: $\frac{\omega Chord}{2q_\infty} = 0.212$
- Pitching Amplitude: ±1.02º

We'll show the comparisons of:

- Dual-time stepping scheme (BDF)
- Pure ADI-BDF
- Hybrid scheme

Inviscid Mach Contour

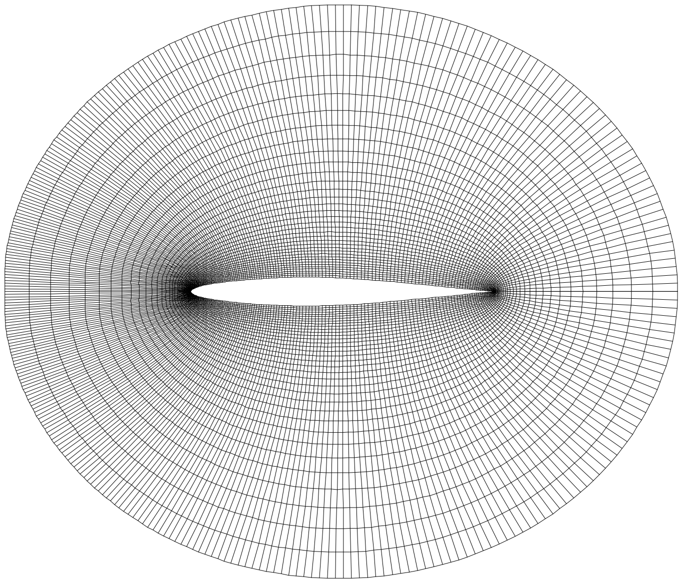
- PSTEP: 36
- NCYC: 4
- SCHEME: Hybrid



2D Airfoil Testcase

- Reasons for choosing this particular testcase:
  - Well documented case with detailed experimental results.
  - A large amount of CFD data are available. Comparison with other numerical schemes possible.
  - Size of the problem is not too large, enabling a large number of computational experiments.

Inviscid 2D Airfoil Testcase

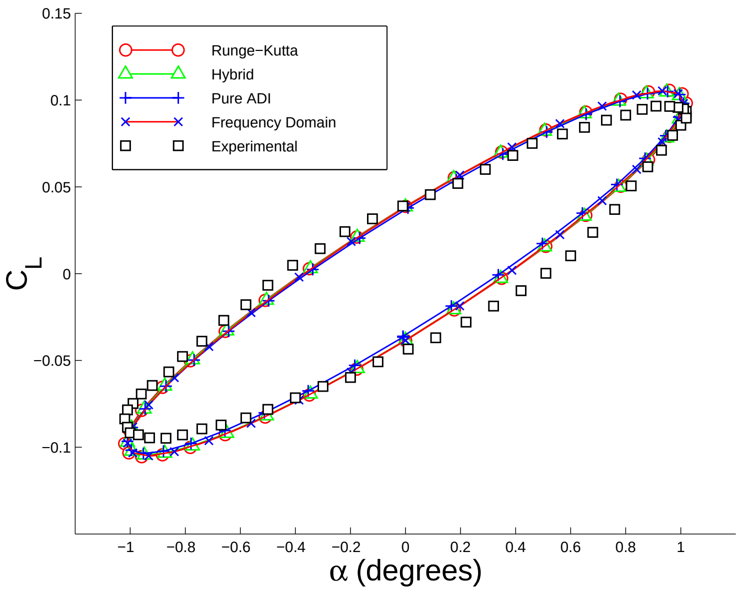


NACA 64A010 Airfoil
Mesh Size: 160X32

- Inner part of a grid obtained via conformal mapping. The grid extends to 100 chords.
- Note that the cells at the trailing edge (TE) are very small in comparison with those at mid-chord (MC).

Inviscid 2D Airfoil Testcase

- Comparison of different numerical schemes:

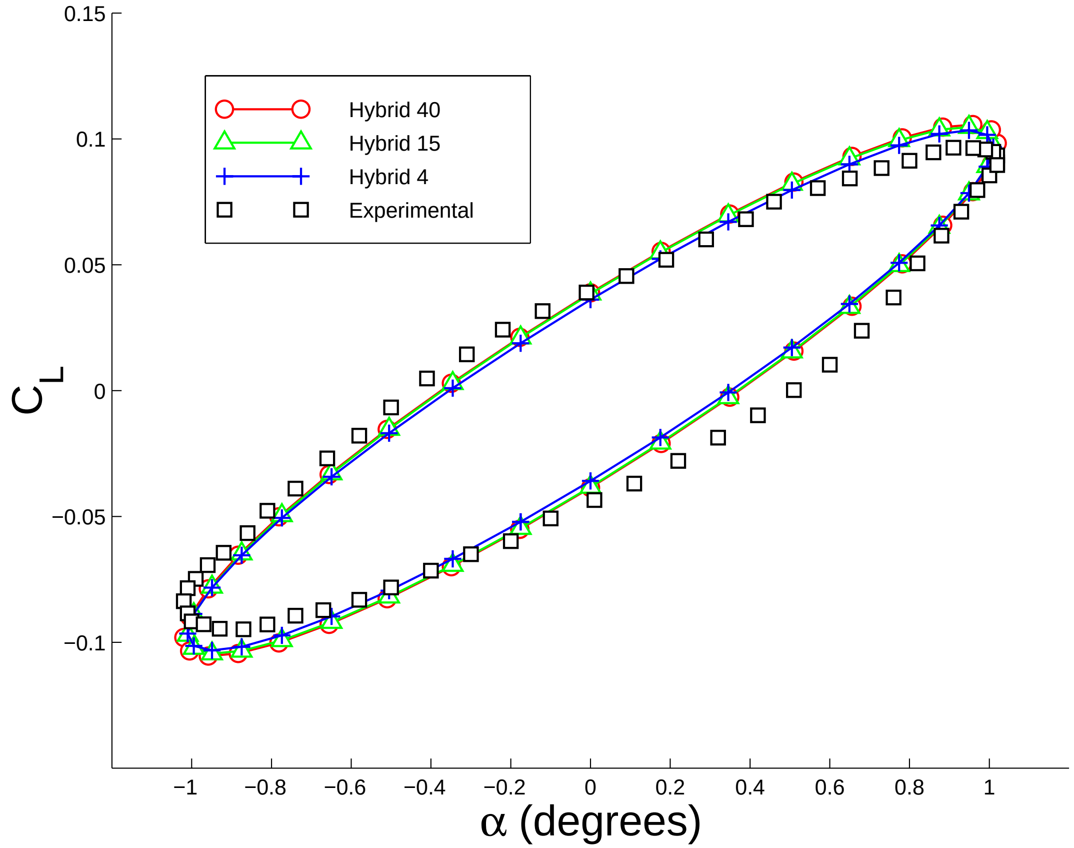


Note that the plots overlap for different numerical schemes.

Scheme	CFL(TE)	CFL(MC)	Iterations	Time Steps
Dual Time Stepping	1764	74	15	36
Hybrid Scheme	1764	74	15	36
Pure ADI-BDF	85	4	1	720

Inviscid 2D Airfoil Testcase

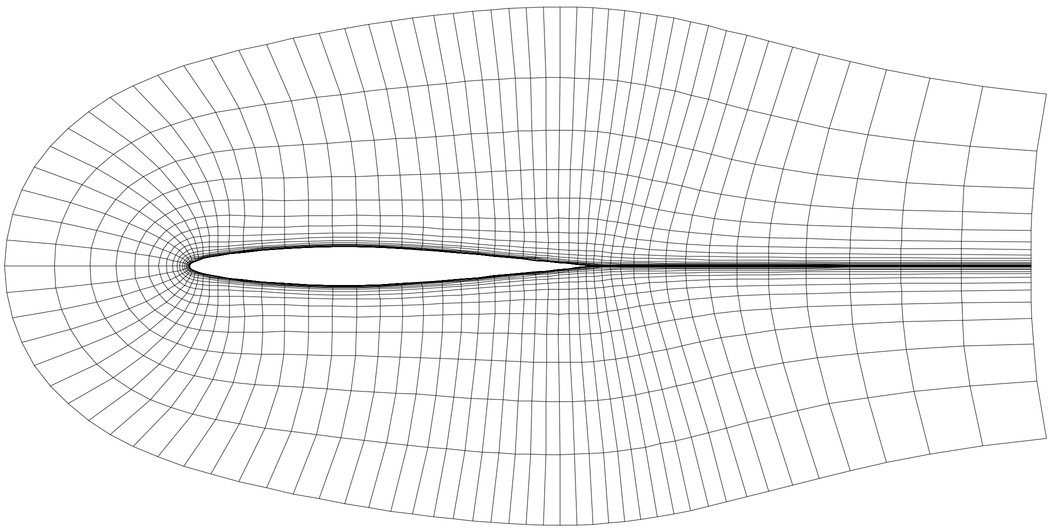
- Comparison of hybrid scheme with different number of iteratio



Inviscid 2D Airfoil Testcase

Scheme	CFL(TE)	CFL(MC)	Inner Iterations	Time Steps	Total Work
Dual Time Stepping	1764	74	15	36	540
Hybrid Scheme	1764	74	15	36	540
Hybrid Scheme	1764	74	4	36	144
Pure ADI-BDF	85	4	1	720	720

Viscous 2D Airfoil Testcase

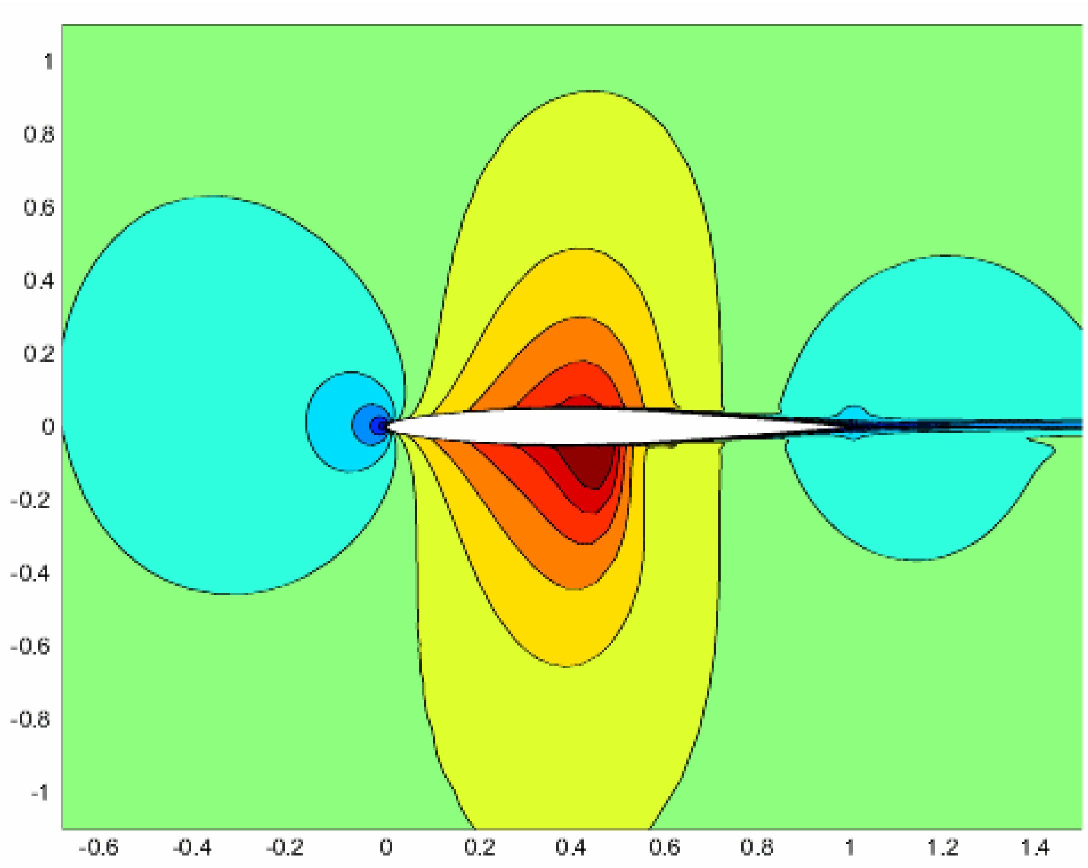


NACA 64A010 Airfoil
Mesh Size: 254X64

- Baldwin-Lomax Turbulence Model.
- Re = 12.56 Million.

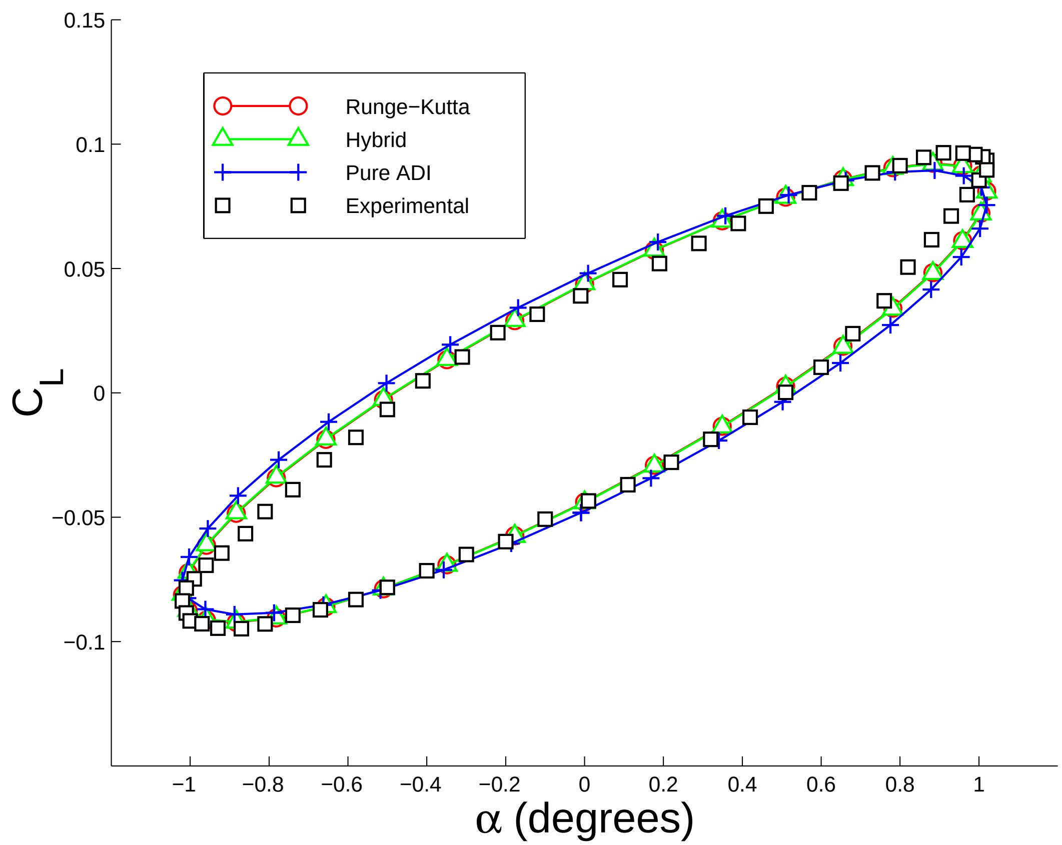
Viscous Mach Contour

- PSTEP: 36
- NCYC: 10
- SCHEME: Hybrid



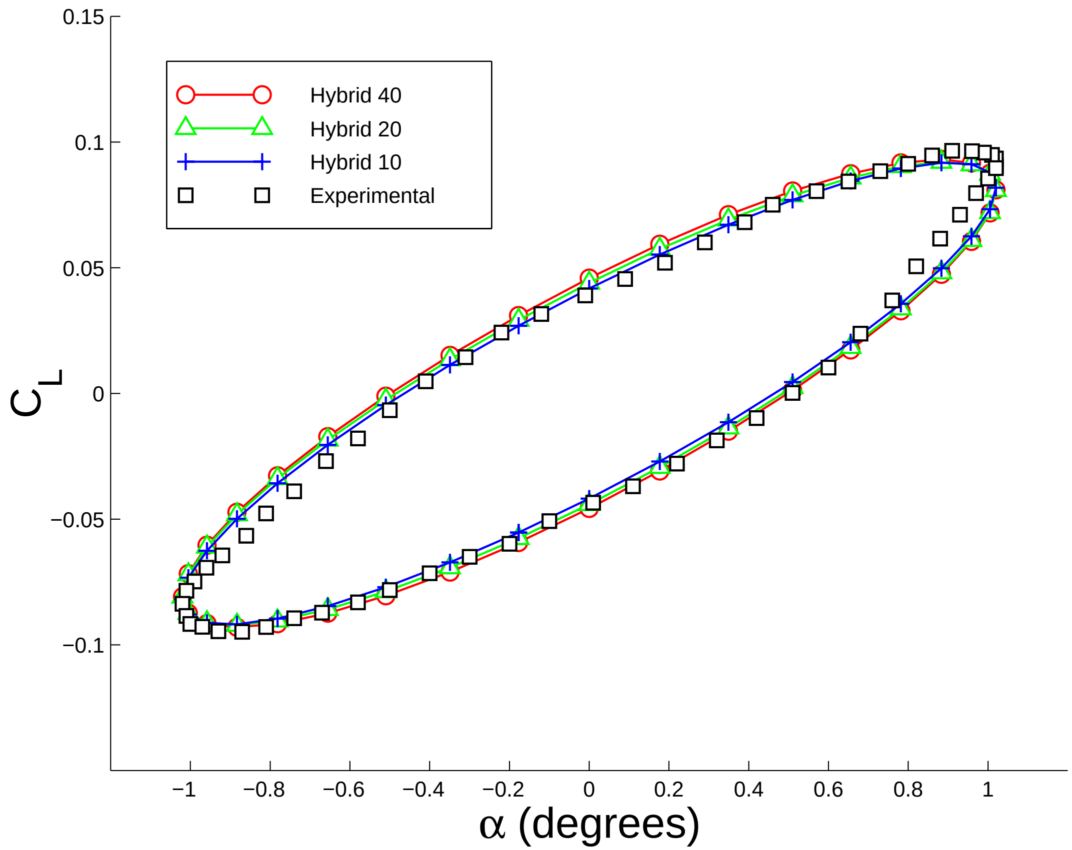
Viscous 2D Airfoil Testcase

- Comparison of different numerical schemes:



Viscous 2D Airfoil Testcase

- Different number of inner iterations for the hybrid scheme:



Viscous 2D Airfoil Testcase

Scheme	Max CFL	Inner Iterations	Time Steps	Total Work
Dual Time Stepping	33500	20	36	720
Hybrid Scheme	33500	20	36	720
Hybrid Scheme	33500	10	36	360
Pure ADI-BDF	1616	1	720	720

Conclusions

- We can obtain second order accuracy without the need to iterate to convergence, using an ADI-BDF step followed by small numbers of dual-time stepping iterations (of the order of 4 or 5 for inviscid, 10 to 15 for viscous calculations).
- The scheme should allow a substantial reduction in the cost of unsteady flow simulations in turbomachinery, for which we currently use 30 inner iterations, perhaps a factor of three. This is significant for calculations which currently require 1.04 million CPU hours.

Future Work

- Further refinement of the hybrid scheme
  - LU-SGS Scheme