1 Flux Splitting, Diffusion and Artificial Dissipation

1.1 What is Diffusion?

Diffusion is the tendency for certain quantity of a medium to spread out evenly. It drives the quantity of any medium from high density locations towards low density locations. It drives in the direction of the opposite of the gradient. In other words, it smoothes scalar or vector fields.

1.2 What is Artificial Diffusion

Artificial diffusion is introduced by adding additional terms to the flux of the discretized equations. In our example, the governing convective (non-dissipated) equations from Euler in one dimension is

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0 \tag{1.1}$$

which can be written in the flux form

$$w^{n+1} = w^n - \Delta t \left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}} \right) \tag{1.2}$$

The fluxes $h_{i+\frac{1}{2}}$ contains the convection part and the diffusion part, where in the 1st order scalar diffusion setting, the artificial dissipation is usually defined as below

$$\begin{aligned} h_{i+\frac{1}{2}} &= f_{i+\frac{1}{2}} - d_{i+\frac{1}{2}} \\ &= \overbrace{\frac{f_{i+1} + f_i}{2}}^{convection} - \underbrace{\epsilon^{(2)} \left(w_{i+1} - w_i \right)}_{diffusion} \end{aligned} \tag{1.3}$$

1.3 Equivalence Between Artificial Diffusion and Flux Splitting

The non-conservative form of the equation is

$$\frac{\partial w}{\partial t} + A \frac{\partial w}{\partial x} = 0 \tag{1.4}$$

$$\tag{1.5}$$

where the Jacobian is

$$A = \frac{\partial f(w)}{\partial x} \tag{1.6}$$

$$A = N^{-1} \Lambda N \tag{1.7}$$

Flux split is done by splitting the Jacobians.

$$A^+ = N^{-1}\Lambda^+ N \tag{1.8}$$
$$A^- = N^{-1}\Lambda^- N \tag{1.9}$$

this is equivalent to below

$$A^\pm = \frac{A \pm 2\epsilon I}{2} \tag{1.10}$$

Using second order diffusion without active pressure sensor switches,

$$h_{j+\frac{1}{2}} = \frac{1}{2}\left(f_{j+1} + f_j\right) - \epsilon^{(2)}\left(w_{j+1} + w_j\right) \tag{1.11}$$

therefore,

$$\begin{aligned}\frac{\partial h_{j+\frac{1}{2}}}{\partial w_j} &= \frac{1}{2}\frac{\partial f_j}{\partial w_j} + \epsilon^{(2)} I \\ &= \frac{1}{2}\left(A_j + 2\epsilon^{(2)} I\right) \\ &= A_j^+ \end{aligned} \tag{1.12}$$

and

$$\begin{aligned}\frac{\partial h_{j+\frac{1}{2}}}{\partial w_{j+1}} &= \frac{1}{2}\frac{\partial f_{j+1}}{\partial w_{j+1}} - \epsilon^{(2)} I \\ &= \frac{1}{2}\left(A_{j+1} - 2\epsilon^{(2)} I\right) \\ &= A_{j+1}^- \end{aligned} \tag{1.13}$$

similarly,

$$\frac{\partial h_{j-\frac{1}{2}}}{\partial w_j} = A_j^- \tag{1.14}$$

and

$$\frac{\partial h_{j-\frac{1}{2}}}{\partial w_{j-1}} = A_{j-1}^+ \tag{1.15}$$

Overall, for the ADI type methods, where the tridiagonal Jacobian matrices looks like below,

$$\begin{bmatrix} D & C & 0 & 0 & 0 & 0 \\ A & D & C & 0 & 0 & 0 \\ 0 & A & D & C & 0 & 0 \\ 0 & 0 & A & D & C & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \\ \Delta w_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots \\ \text{R.H.S.} \\ \cdots \end{bmatrix} \tag{1.16}$$

the diffusion of the difference of the fluxes at $j+1$ and j written in first order expansion of Taylor series,

$$
\begin{aligned}
&\overbrace{\frac{\partial\left(h_{j+\frac{1}{2}}-h_{j-\frac{1}{2}}\right)}{\partial w_{j-1}}}^{A}\Delta w_{j-1}+\overbrace{\frac{\partial\left(h_{j+\frac{1}{2}}-h_{j-\frac{1}{2}}\right)}{\partial w_{j}}}^{D}\Delta w_{j}+\overbrace{\frac{\partial\left(h_{j+\frac{1}{2}}-h_{j-\frac{1}{2}}\right)}{\partial w_{j+1}}}^{C}\Delta w_{j+1}\\
&\qquad = A^{-}_{j+1}\Delta w_{j+1}-A^{-}_{j}\Delta w_{j}+A^{+}_{j}\Delta w_{j}-A^{+}_{j-1}\Delta w_{j-1}\\
&\qquad = \left(D^{+}_{x}A^{-}+D^{-}_{x}A^{+}\right)\Delta w\\
&\qquad = \underbrace{\left(-A^{+}_{j-1}\right)}_{A}\Delta w_{j-1}+\underbrace{\left(A^{+}_{j}-A^{-}_{j}\right)}_{D}\Delta w_{j}+\underbrace{\left(A^{-}_{j+1}\right)}_{C}\Delta w_{j+1} \qquad (1.17)
\end{aligned}
$$

this results in

$$
\begin{bmatrix}
-A_2^- + A_2^+ & A_3^- & 0 & 0 & 0 & 0 \\
-A_2^+ & -A_3^- + A_3^+ & A_4^- & 0 & 0 & 0 \\
0 & -A_3^+ & -A_4^- + A_4^+ & A_5^- & 0 & 0 \\
0 & 0 & -A_4^+ & -A_5^- + A_5^+ & A_6^- & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
\Delta w_2 \\ \Delta w_3 \\ \Delta w_4 \\ \Delta w_5 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\cdots \\ \text{R.H.S.} \\ \cdots
\end{bmatrix}
\qquad (1.18)
$$

Therefore, flux splitting is equivalent to added diffusion.