Eigenvalues and Eigenvectors for the Gas Dynamic Equations

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Apr 9, 2001

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1 Three Dimensional Inviscid Flow

1.1 General Decomposition

The Euler Equations for the three dimensional flow of an inviscid gas can be written as

$$\frac{d}{dt}\int_{\Omega} w d\mathcal{V} + \int_{\partial\Omega} \left(F_x dS_x + F_y dS_y + F_z dS_z\right) = 0 \tag{1.1}$$

where w is the state vector, F_x, F_y and F_z are the flux vectors, and dS_x, dS_y and dS_z are the projection of the surface element in the coordinate directions x, y and z. Let u, v and w be the velocity components and ρ, p, E and H be the density, pressure, total energy and total enthalpy. Then

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, F_x = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ \rho uH \end{pmatrix}, F_y = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ \rho vH \end{pmatrix}, F_z = \begin{pmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ \rho wH \end{pmatrix} \tag{1.2a}$$

also

$$p = (\gamma - 1)\,\rho\left(E - \frac{q^2}{2}\right) \tag{1.2b}$$

$$H = E + \frac{p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{q^2}{2} \tag{1.2c}$$

where q is the speed and c is the speed of sound

$$q^2 = u^2 + v^2 + w^2 \tag{1.2d}$$

$$c^2 = \frac{\gamma p}{\rho} \tag{1.2e}$$

When the flow is smooth, it can be represented by the quasilinear differential equation

$$\frac{\partial w}{\partial t} + A_x\frac{\partial w}{\partial x} + A_y\frac{\partial w}{\partial y} + A_z\frac{\partial w}{\partial z} = 0 \tag{1.3}$$

where A_x, A_y and A_z are the Jacobian matrices

$$A_x = \frac{\partial F_x}{\partial w}, \quad A_y = \frac{\partial F_y}{\partial w}, \quad A_z = \frac{\partial F_z}{\partial w} \tag{1.4}$$

Under a change of variables to a new state vector $\tilde{w}$, equation (1.3) is transformed to

$$\frac{\partial \tilde{w}}{\partial t} + \tilde{A}_x \frac{\partial \tilde{w}}{\partial x} + \tilde{A}_y \frac{\partial \tilde{w}}{\partial y} + \tilde{A}_z \frac{\partial \tilde{w}}{\partial z} = 0 \tag{1.5}$$

where

$$A_x = \tilde{M}\tilde{A}_x\tilde{M}^{-1}, \quad A_y = \tilde{M}\tilde{A}_y\tilde{M}^{-1}, \quad A_z = \tilde{M}\tilde{A}_z\tilde{M}^{-1} \tag{1.6}$$

and

$$\tilde{M} = \frac{\partial w}{\partial \tilde{w}} \tag{1.7}$$

The finite volume discretization requires the evaluation of the flux through a face with vector area S,

$$F = F_x S_x + F_y S_y + F_z S_z \tag{1.8}$$

the corresponding Jacobian matrix is

$$A = \frac{\partial F}{\partial w} = S_x A_x + S_y A_y + S_z A_z \tag{1.9}$$

It is convenient to represent A in terms of the Jacobian matrices $\tilde{A}_x$, $\tilde{A}_y$ and $\tilde{A}_z$ which are obtained under a transformation of the dependant variables, which may be chosen to give these matrices a sparse form. Then

$$A = \tilde{M}\tilde{A}\tilde{M}^{-1} \tag{1.10}$$

where

$$\tilde{A} = S_x\tilde{A}_x + S_y\tilde{A}_y + S_z\tilde{A}_z \tag{1.11}$$

Now if the columns of $\tilde{R}$ are right eigenvectors of $\tilde{A}$, and the rows of $\tilde{R}^{-1}$ are left eigenvectors, $\tilde{A}$ can be decomposed

$$\tilde{A} = \tilde{R}\Lambda\tilde{R}^{-1} \tag{1.12}$$

where Λ is a diagonal matrix containing the eigenvalues. Correspondingly

$$A = R\Lambda R^{-1} \tag{1.13}$$

where

$$R = \tilde{M}\tilde{R}, \quad R^{-1} = \tilde{R}^{-1}\tilde{M}^{-1} \tag{1.14}$$

Introduce the primitive variables

$$\tilde{w} = \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix} \tag{1.15}$$

one finds that

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 \\ v & 0 & \rho & 0 & 0 \\ w & 0 & 0 & \rho & 0 \\ \frac{q^2}{2} & \rho u & \rho v & \rho w & \frac{1}{(\gamma-1)} \end{pmatrix}$$

$$\tilde{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)q^2}{2} & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-1)w & (\gamma-1) \end{pmatrix} \tag{1.16}$$

and

$$\tilde{A} = \begin{pmatrix} Q & S_x\rho & S_y\rho & S_z\rho & 0 \\ 0 & Q & 0 & 0 & \frac{S_x}{\rho} \\ 0 & 0 & Q & 0 & \frac{S_y}{\rho} \\ 0 & 0 & 0 & Q & \frac{S_z}{\rho} \\ 0 & S_x\rho c^2 & S_y\rho c^2 & S_z\rho c^2 & Q \end{pmatrix} \tag{1.17}$$

where

$$Q = S_x u + S_y v + S_z w \tag{1.18}$$

The density can be eliminated by setting

$$\tilde{M} = \hat{M} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.19}$$

Then

$$A = \hat{M}\hat{A}\hat{M}^{-1} \tag{1.20}$$

where

$$
\hat{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ u & 1 & 0 & 0 & 0 \\ v & 0 & 1 & 0 & 0 \\ w & 0 & 0 & 1 & 0 \\ \frac{q^2}{2} & u & v & w & \frac{1}{(\gamma-1)} \end{pmatrix}
$$

$$
\hat{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -u & 1 & 0 & 0 & 0 \\ -v & 0 & 1 & 0 & 0 \\ -w & 0 & 0 & 1 & 0 \\ \frac{(\gamma-1)q^2}{2} & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-1)w & (\gamma-1) \end{pmatrix} \tag{1.21}
$$

and

$$
\hat{A} = \begin{pmatrix} Q & S_x & S_y & S_z & 0 \\ 0 & Q & 0 & 0 & S_x \\ 0 & 0 & Q & 0 & S_y \\ 0 & 0 & 0 & Q & S_z \\ 0 & S_x c^2 & S_y c^2 & S_z c^2 & Q \end{pmatrix} \tag{1.22}
$$

Let S be the magnitude of the face areas,

$$
S^2 = S_x^2 + S_y^2 + S_z^2 \tag{1.23}
$$

and n_x, n_y and n_z be the components of the unit normal

$$
n_x = \frac{S_x}{S}, \quad n_y = \frac{S_y}{S}, \quad n_z = \frac{S_z}{S} \tag{1.24}
$$

Then the eigenvalues of $\hat{A}$ are

$$
Q, \quad Q, \quad Q, \quad Q + cS, \quad Q - cS \tag{1.25}
$$

Every element of $\hat{A}$ is expressed in terms of velocities and areas. Since the eigenvectors can be scaled aritrarily, the scale factors can be chosen so that the eigenvectors are expressed in terms of velocities and the elements of the normal. Moreover, the eigenvectors corresponding to the eigenvalue Q are not unique, since any linear combination of these eigenvectors is another eigenvector for the same eigenvalue. Three independent eigenvectors corresponding to the eigenvalue Q are

$$
\hat{r}_1 = \begin{pmatrix} n_x \\ 0 \\ n_z \\ -n_y \\ 0 \end{pmatrix}, \hat{r}_2 = \begin{pmatrix} n_y \\ -n_z \\ 0 \\ n_x \\ 0 \end{pmatrix}, \hat{r}_3 = \begin{pmatrix} n_z \\ n_y \\ -n_x \\ 0 \\ 0 \end{pmatrix} \tag{1.26}
$$

Eigenvectors corresponding to the eigenvalues $Q + cS$ and $Q - cS$ are

$$\hat{r}_4 = \begin{pmatrix} \frac{1}{c} \\ n_x \\ n_y \\ n_z \\ c \end{pmatrix} \quad and \quad \hat{r}_5 = \begin{pmatrix} \frac{1}{c} \\ -n_x \\ -n_y \\ -n_z \\ c \end{pmatrix} \tag{1.27}$$

The linear combination

$$r^* = n_x\hat{r}_1 + n_y\hat{r}_2 + n_z\hat{r}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.28}$$

is another eigenvector corresponding to the eigenvalue Q. Therefore $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$ may be replaced by $\hat{r}_1 - \alpha_1 r^*$, $\hat{r}_2 - \alpha_2 r^*$ and $\hat{r}_3 - \alpha_3 r^*$ where α_1, α_2 and α_3 are arbitrary. Using this procedure to rescale the first elements of $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$, the matrix of right eigenvectors of $\hat{A}$ can be expressed as

$$\hat{R} = \begin{pmatrix} \frac{1}{c} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} n_x & n_y & n_z & 1 & 1 \\ 0 & -n_z & n_y & n_x & -n_x \\ n_z & 0 & -n_x & n_y & -n_y \\ -n_y & n_x & 0 & n_z & -n_z \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{1.29}$$

Now define

$$M = \hat{M}D, \quad M^{-1} = D^{-1}\hat{M}^{-1} \tag{1.30}$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & c^2 \end{pmatrix}, \quad D^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c^2} \end{pmatrix} \tag{1.31}$$

Then,

$$\hat{A} = DN\Lambda N^{-1}D^{-1}, \quad A = MN\Lambda N^{-1}M^{-1} \tag{1.32}$$

where

$$\Lambda = \begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 0 & 0 & Q+cS & 0 \\ 0 & 0 & 0 & 0 & Q-cS \end{pmatrix} \tag{1.33}$$

$$
M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ u & c & 0 & 0 & 0 \\ v & 0 & c & 0 & 0 \\ w & 0 & 0 & c & 0 \\ \frac{q^2}{2} & uc & vc & wc & \frac{c^2}{(\gamma-1)} \end{pmatrix}
$$

$$
M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{u}{c} & \frac{1}{c} & 0 & 0 & 0 \\ -\frac{v}{c} & 0 & \frac{1}{c} & 0 & 0 \\ -\frac{w}{c} & 0 & 0 & \frac{1}{c} & 0 \\ \frac{(\gamma-1)q^2}{2c^2} & \frac{-(\gamma-1)u}{c^2} & \frac{-(\gamma-1)v}{c^2} & \frac{-(\gamma-1)w}{c^2} & \frac{(\gamma-1)}{c^2} \end{pmatrix} \tag{1.34}
$$

and

$$
N = \begin{pmatrix} n_x & n_y & n_z & 1 & 1 \\ 0 & -n_z & n_y & n_x & -n_x \\ n_z & 0 & -n_x & n_y & -n_y \\ -n_y & n_x & 0 & n_z & -n_z \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}
$$

$$
N^{-1} = \begin{pmatrix} n_x & 0 & n_z & -n_y & -n_x \\ n_y & -n_z & 0 & n_x & -n_y \\ n_z & n_y & -n_x & 0 & -n_z \\ 0 & \frac{n_x}{2} & \frac{n_y}{2} & \frac{n_z}{2} & \frac{1}{2} \\ 0 & \frac{-n_x}{2} & \frac{-n_y}{2} & \frac{-n_z}{2} & \frac{1}{2} \end{pmatrix} \tag{1.35}
$$

Also the columns of MN are right eigenvectors of A. Set

$$
v_0 = \begin{pmatrix} 1 \\ u \\ v \\ w \\ \frac{q^2}{2} \end{pmatrix} \quad v_1 = \begin{pmatrix} 0 \\ 0 \\ n_z \\ -n_y \\ vn_z - wn_y \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ -n_z \\ 0 \\ n_x \\ wn_x - un_z \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ n_y \\ -n_x \\ 0 \\ un_y - vn_x \end{pmatrix} \tag{1.36}
$$

Then the eigenvectors corresponding to the eigenvalue Q are

$$
r_1 = n_x v_0 + v_1, \quad r_2 = n_y v_0 + v_2, \quad r_3 = n_z v_0 + v_3 \tag{1.37}
$$

Here v_0 represents an entropy wave and v_k represents vorticity waves. Also, let q_n denote the normal velocity $\frac{Q}{S}$, and set

$$
v_4 = \begin{pmatrix} 1 \\ u \\ v \\ w \\ H \end{pmatrix} \quad v_5 = \begin{pmatrix} 0 \\ n_x \\ n_y \\ n_z \\ q_n \end{pmatrix} \tag{1.38}
$$

Then the last two eigenvectors, corresponding to pressure waves, are

$$r_4 = v_4 + cv_5, \quad r_5 = v_4 - cv_5 \tag{1.39}$$

1.2 Symmetric Decomposition

The differential variables

$$d\bar{w} = \begin{pmatrix} \frac{dp}{\rho c} \\ du \\ dv \\ dw \\ dp - c^2 d\rho \end{pmatrix} \tag{1.40}$$

lead to the transformation matrix

$$\begin{aligned} L^{-1} &= \frac{\partial \bar{w}}{\partial \tilde{w}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\rho c} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -c^2 & 0 & 0 & 0 & 1 \end{pmatrix} \\ L &= \frac{\partial \tilde{w}}{\partial \bar{w}} = \begin{pmatrix} \frac{\rho}{c} & 0 & 0 & 0 & -\frac{1}{c^2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \rho c & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{1.41}$$

This reduces the Jacobian matrix to the symmetric form

$$\bar{A} = L^{-1}\tilde{A}L = \begin{pmatrix} Q & S_x c & S_y c & S_z c & 0 \\ S_x c & Q & 0 & 0 & 0 \\ S_y c & 0 & Q & 0 & 0 \\ S_z c & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & Q \end{pmatrix} \tag{1.42}$$

Since the transformation is independent of S_x, S_y and S_z, these variables reduce (1.3) to a form in which all three Jacobian matrices are symmetric. Also

$$dp - c^2 d\rho = p ds \tag{1.43}$$

where s is the entropy $log(\frac{p}{\rho^\gamma})$ Thus the last of these variables represents entropy, which is constant along streamlines in regions of the flow not containing shock waves. If the flow is isentropic, the first variable is $\int \frac{dp}{\rho c} = \frac{2c}{\gamma-1}$, and in one dimensional flow the equations for the Riemann invariants $u \pm \frac{2c}{\gamma-1}$ are recovered by combining the first two of equations.

Now

$$A = \tilde{M} L \bar{A} L^{-1} \tilde{M}^{-1} \tag{1.44}$$

where

$$\tilde{M}L = \begin{pmatrix} \frac{\rho}{c} & 0 & 0 & 0 & -\frac{1}{c^2} \\ \frac{\rho u}{c} & \rho & 0 & 0 & -\frac{u}{c^2} \\ \frac{\rho v}{c} & 0 & \rho & 0 & -\frac{v}{c^2} \\ \frac{\rho w}{c} & 0 & 0 & \rho & -\frac{w}{c^2} \\ \frac{\rho H}{c} & \rho u & \rho v & \rho w & -\frac{q^2}{2c^2} \end{pmatrix} \tag{1.45}$$

Since A is related to $\bar{A}$ by a similarity transformation, it has the same eiganvalues. Let S be the magnitude of the face area

$$S^2 = S_x^2 + S_y^2 + S_z^2 \tag{1.46}$$

and n_x, n_y and n_z be the components of the unit normal

$$n_x = \frac{S_x}{S}, \quad n_y = \frac{S_y}{S}, \quad n_z = \frac{S_z}{S} \tag{1.47}$$

Then the eigenvalues of $\bar{A}$ are Q, Q, Q, $Q + cS$ and $Q - cS$. Also the corresponding set of right and left eigenvectors are the columns and rows of

$$\begin{aligned} N &= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & -n_z & n_y & n_x & -n_x \\ n_z & 0 & -n_x & n_y & -n_y \\ -n_y & n_x & 0 & n_z & -n_z \\ n_x & n_y & n_z & 0 & 0 \end{pmatrix} \\ N^{-1} &= \begin{pmatrix} 0 & 0 & n_z & -n_y & n_x \\ 0 & -n_z & 0 & n_x & n_y \\ 0 & n_y & -n_x & 0 & n_z \\ \frac{1}{2} & \frac{n_x}{2} & \frac{n_y}{2} & \frac{n_z}{2} & 0 \\ \frac{1}{2} & -\frac{n_x}{2} & -\frac{n_y}{2} & -\frac{n_z}{2} & 0 \end{pmatrix} \end{aligned} \tag{1.48}$$

Then

$$\bar{A} = \bar{N}\Lambda\bar{N}^{-1} \tag{1.49}$$

where

$$\Lambda = \begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 0 & 0 & Q + cS & 0 \\ 0 & 0 & 0 & 0 & Q - cS \end{pmatrix} \tag{1.50}$$

Define

$$\bar{D} = \begin{pmatrix} \frac{\rho}{c} & 0 & 0 & 0 & 0 \\ 0 & \frac{\rho}{c} & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho}{c} & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho}{c} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{c^2} \end{pmatrix}$$

$$\bar{D}^{-1} = \begin{pmatrix} \frac{c}{\rho} & 0 & 0 & 0 & 0 \\ 0 & \frac{c}{\rho} & 0 & 0 & 0 \\ 0 & 0 & \frac{c}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{c}{\rho} & 0 \\ 0 & 0 & 0 & 0 & -c^2 \end{pmatrix} \tag{1.51}$$

and

$$\bar{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ u & c & 0 & 0 & u \\ v & 0 & c & 0 & v \\ w & 0 & 0 & c & w \\ H & uc & vc & wc & \frac{q^2}{2} \end{pmatrix}$$

$$\bar{M}^{-1} = \begin{pmatrix} \frac{(\gamma-1)q^2}{2c^2} & -\frac{(\gamma-1)u}{c^2} & -\frac{(\gamma-1)v}{c^2} & -\frac{(\gamma-1)w}{c^2} & \frac{(\gamma-1)}{c^2} \\ -\frac{u}{c} & \frac{1}{c} & 0 & 0 & 0 \\ -\frac{v}{c} & 0 & \frac{1}{c} & 0 & 0 \\ -\frac{w}{c} & 0 & 0 & \frac{1}{c} & 0 \\ 1-\frac{(\gamma-1)q^2}{2c^2} & \frac{(\gamma-1)u}{c^2} & \frac{(\gamma-1)v}{c^2} & \frac{(\gamma-1)w}{c^2} & -\frac{(\gamma-1)}{c^2} \end{pmatrix} \tag{1.52}$$

Then

$$\bar{M}\bar{D} = \tilde{M}L \tag{1.53}$$

and

$$\bar{D}\bar{A}\bar{D}^{-1} = \bar{A} \tag{1.54}$$

Therefore

$$A = \bar{M}\bar{A}\bar{M}^{-1} = \bar{M}\bar{N}\Lambda\bar{N}^{-1}\bar{M}^{-1} \tag{1.55}$$

and the columns of $\bar{M}\bar{N}$ are right eigenvectors of A.

$$\tag{1.56}$$

The relation between the two representations of A is

$$\bar{M} = MP, \quad \bar{N} = P^{-1}N, \quad \hat{A} = DP\bar{A}P^{-1}D^{-1} \tag{1.57}$$

where

$$
\begin{aligned}
P &= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
P^{-1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
\tag{1.58}
$$

2 Two Dimensional E-Characteristic Eigenvectors

$$A = \tilde{M}\tilde{A}\tilde{M}^{-1} = \hat{M}\hat{A}\hat{M}^{-1} \tag{2.1}$$

$$\begin{aligned}
\tilde{M} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{q^2}{2} & \rho u & \rho v & \frac{1}{(\gamma-1)} \end{pmatrix} \\
\tilde{M}^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)q^2}{2} & -(\gamma-1)u & -(\gamma-1)v & (\gamma-1) \end{pmatrix}
\end{aligned} \tag{2.2}$$

$$\tilde{A} = \begin{pmatrix} Q_R & S_x\rho & S_y\rho & 0 \\ 0 & Q_R & 0 & \frac{S_x}{\rho} \\ 0 & 0 & Q_R & \frac{S_y}{\rho} \\ 0 & S_x\rho c^2 & S_y\rho c^2 & Q_R \end{pmatrix}, \quad Q_R = uS_x + vS_y - Q_{mesh} \tag{2.3}$$

$$\begin{aligned}
\hat{M} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ v & 0 & 1 & 0 \\ \frac{q^2}{2} & u & v & \frac{1}{(\gamma-1)} \end{pmatrix} \\
\hat{M}^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -u & 1 & 0 & 0 \\ -v & 0 & 1 & 0 \\ \frac{(\gamma-1)q^2}{2} & -(\gamma-1)u & -(\gamma-1)v & (\gamma-1) \end{pmatrix}
\end{aligned} \tag{2.4}$$

$$\hat{A} = \begin{pmatrix} Q_R & S_x & S_y & 0 \\ 0 & Q_R & 0 & S_x \\ 0 & 0 & Q_R & S_y \\ 0 & S_x c^2 & S_y c^2 & Q_R \end{pmatrix} = P\tilde{A}P^{-1}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

where $Q_R = Q - Q_{mesh}$
Also,

$$\tilde{A} = \tilde{T}\Lambda\tilde{T}^{-1}, \quad \hat{A} = \hat{T}\Lambda\hat{T}^{-1} = DN\Lambda N^{-1}D^{-1} \tag{2.6}$$

$$\Lambda = \begin{pmatrix} Q_R & 0 & 0 & 0 \\ 0 & Q_R & 0 & 0 \\ 0 & 0 & Q_R + cS & 0 \\ 0 & 0 & 0 & Q - cS \end{pmatrix} \tag{2.7}$$

$$
\begin{aligned}
\tilde{T} &= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -\frac{c}{\rho}n_y & \frac{c}{\rho}n_x & -\frac{c}{\rho}n_x \\ 0 & \frac{c}{\rho}n_x & \frac{c}{\rho}n_y & -\frac{c}{\rho}n_y \\ 0 & 0 & c^2 & c^2 \end{pmatrix} \\
\tilde{T}^{-1} &= \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{c^2} \\ 0 & -\frac{\rho}{c}n_y & \frac{\rho}{c}n_x & 0 \\ 0 & \frac{\rho}{2c}n_x & \frac{\rho}{2c}n_y & \frac{1}{2c^2} \\ 0 & -\frac{\rho}{2c}n_x & -\frac{\rho}{2c}n_y & \frac{1}{2c^2} \end{pmatrix}
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
\hat{T} &= DN = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -n_y & n_x & -n_x \\ 0 & n_x & n_y & -n_y \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
\hat{T}^{-1} &= N^{-1}D^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -n_y & n_x & 0 \\ 0 & \frac{n_x}{2} & \frac{n_y}{2} & \frac{1}{2} \\ 0 & -\frac{n_x}{2} & -\frac{n_y}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{c} & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{c^2} \end{pmatrix}
\end{aligned} \tag{2.9}
$$

3 Appendix

$$\frac{\partial w}{\partial t} + A_x \frac{\partial w}{\partial x} + A_y \frac{\partial w}{\partial y} = 0 \tag{3.1}$$

$$\frac{\partial \tilde{w}}{\partial w}\frac{\partial w}{\partial t} + A_x \frac{\partial \tilde{w}}{\partial w}\frac{\partial w}{\partial x} + A_y \frac{\partial \tilde{w}}{\partial w}\frac{\partial w}{\partial y} = 0 \tag{3.2}$$

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\partial F}{\partial w} + \frac{\partial \tilde{w}}{\partial x} + \frac{\partial G}{\partial w} + \frac{\partial \tilde{w}}{\partial y} = 0 \tag{3.3}$$

$$\frac{\partial w}{\partial \tilde{w}}\frac{\partial \tilde{w}}{\partial t} + \frac{\partial w}{\partial \tilde{w}}\frac{\partial F}{\partial w} + \frac{\partial \tilde{w}}{\partial x} + \frac{\partial w}{\partial \tilde{w}}\frac{\partial G}{\partial w} + \frac{\partial \tilde{w}}{\partial y} = 0 \tag{3.4}$$

$$\frac{\partial w}{\partial t} + \frac{\partial F}{\partial \tilde{w}} + \frac{\partial \tilde{w}}{\partial x} + \frac{\partial G}{\partial \tilde{w}} + \frac{\partial \tilde{w}}{\partial y} = 0 \tag{3.5}$$