Advanced Computational Fluid Dynamics
AA215A Lecture 1
Vector and Function Spaces

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Contents

Lecture 1

Vector and Function Spaces

1.1 Basic concepts

The well known concepts of Euclidean space can be generalized to n dimensional vectors and also to functions. These concepts are extremely useful in the analysis of numerical methods and are briefly reviewed in this appendix. The idea of function space was one of the great achievements of nineteenth century mathematics, particularly due to Banach and Hilbert. Hilbert was the first to introduce spaces with abstract inner product. As a preliminary, it is useful to derive some basic inequalities due to Jensen, Hölder, Cauchy and Schwarz.

1.2 Convex and concave functions: Jensen's theorem

A function of $f(x)$ of a real variable x is convex if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2), \tag{1.1}$$

and concave if

$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2), \tag{1.2}$$

whenever $0 < t < 1$. It is strictly convex or concave if equality in these expressions implies $x_1 = x_2$.

If f is twice differentiable, it is convex if $f'' \geq 0$ and concave if $f'' \leq 0$, and strictly convex or concave if $f'' > 0$ or $f'' < 0$.

Jensen's theorem:

If $f(x)$ is a concave function,

$$\sum_{i=1}^{n} t_i f(x_i) \leq f\left(\sum_{i=1}^{n} t_i x_i\right), \tag{1.3}$$

whenever $t_1, ..., t_n \in (0,1)$ and $\sum_{i=1}^{n} t_i = 1$.

This may be proved by induction. It is true for $n = 2$ by the definition of a concave function. Suppose $n \geq 3$ and the assertion holds for smaller values of n. Then, for $i = 2, ..., n$, set $t_i' =$

$t_i/(1-t_1)$, so that $\sum_{i=2}^n t_i' = 1$. Then,

$$\begin{aligned}\sum_{i=1}^n t_i f(x_1) &= t_1 f(x_1) + (1-t_1)\sum_{i=2}^n t_i' f(x_i) \\ &\leq t_1 f(x_1) + (1-t_1) f\left(\sum_{i=2}^n t_i' x_i\right) \\ &\leq f\left(t_1 x_1 + (1-t_1)\sum_{i=2}^n t_i' x_i\right) \\ &= f\left(\sum_{i=1}^n t_i x_i\right)\end{aligned}$$

1.3 The generalized AM-GM inequality

The geometric mean (GM) of n real numbers does not exceed the arithmetic mean (AM). This result follows from Jensen's theorem applied to the function $\log x$, which is strictly concave Thus if $p_1, ..., p_n > 0$ and $\sum_{i=1}^n p_i = 1$

$$\sum_{i=1}^n p_i \log a_i \leq \log\left(\sum_{i=1}^n p_i a_i\right) \tag{1.4}$$

and hence

$$\prod_{i=1}^n a_i^{p_i} \leq \sum_{i=1}^n p_i a_i \tag{1.5}$$

This is known as the generalized AM-GM inequality. Setting $p_i = \frac{1}{n}$, recovers the basic AM-GM inequality

$$\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^n a_i \tag{1.6}$$

1.4 Hölder's inequality and the Cauchy-Schwarz inequality

Suppose $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\sum_{k=1}^n a_k b_k\right| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |b_k|^q\right)^{\frac{1}{q}} \tag{1.7}$$

Set $x_1 = a^p$, $x_2 = b^q$, $p_1 = \frac{1}{p}$, $p_2 = \frac{1}{q}$. By the generalized GM-AM theorem

$$ab = x_1^{p_1} x_2^{p_2} \leq p_1 x_1 + p_2 x_2 = \frac{a^p}{p} + \frac{b^q}{q} \tag{1.8}$$

Take vectors such that

$$\sum_{k=1}^n |a_k|^p = \sum_{k=1}^n |b_k|^q = 1 \tag{1.9}$$

Then

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \sum_{k=1}^{n} |a_k b_k| \leq \sum_{k=1}^{n} \left(\frac{|a_k|^p}{p} + \frac{|b_k|^q}{q}\right) = \frac{1}{p} + \frac{1}{q} = 1 \tag{1.10}$$

When $p = 2$ this reduces to the Cauchy-Schwartz inequality:

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \tag{1.11}$$

This may be proved directly by noting that for any value of ρ

$$\sum_{k=1}^{n} (a_k + \rho b_k)(a_k + \rho b_k) \geq 0 \tag{1.12}$$

and consequently

$$\sum_{k=1}^{n} a_k^2 + 2\rho \left|\sum_{k=1}^{n} a_k b_k\right| + \rho^2 \sum_{k=1}^{n} b_k^2 \geq 0 \tag{1.13}$$

Then setting

$$\rho = -\frac{|\sum_{k=1}^{n} a_k b_k|}{\sum_{k=1}^{n} b_k^2} \tag{1.14}$$

and multiplying by $\sum_{k=1}^{n} b_k^2$

$$\left(\sum_{k=1}^{n} a_k^2\right) \left(\sum_{k=1}^{n} b_k^2\right) \geq \left|\sum_{k=1}^{n} a_k b_k\right| \tag{1.15}$$

1.5 Vector Norms

The size of an n-dimensional vector can be conveniently represented by a variety of measures. Such measures are called norms, which are required to satisfy the following axioms:

To each vector $\mathbf{x}$ assign a number $||\mathbf{x}||$ where

1. $||\mathbf{x}|| \geq 0$
2. $||\alpha\mathbf{x}|| = |\alpha| ||\mathbf{x}||$ for scalar α
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ (triangle inequality)
4. $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

Some widely used norms are

- $||\mathbf{x}||_p = (\sum |x_i|^p)^{\frac{1}{p}}$ for $p = 1, 2, 3, ...\infty$
- $||\mathbf{x}||_1 = \sum |x_i|$
- $||\mathbf{x}||_2 = \left(\sum |x_i|^2\right)^{\frac{1}{2}}$
- $||\mathbf{x}||_\infty = \max_i |x_i|$

Conditions (1) and (2) are evident. For (3)

- $||\mathbf{x}+\mathbf{y}||_1 = \sum |x_i + y_i| \leq \sum (|x_i| + |y_i|) \leq ||\mathbf{x}||_1 + ||\mathbf{y}||_1$
- $||\mathbf{x}+\mathbf{y}||_\infty = \max_i |x_i + y_i| \leq \max_i |x_i| + \max_i |y_i| \leq ||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty$

To verify (3) for $||\mathbf{x}||_2$ use Cauchy-Schwartz inequality

$$|\mathbf{x}^T\mathbf{y}| \leq ||\mathbf{x}||\ ||\mathbf{y}|| \tag{1.16}$$

Then

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||_2 &= \left((\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y})\right)^{\frac{1}{2}} \\ &\leq \left(\mathbf{x}^T\mathbf{x} + 2|\mathbf{y}^T\mathbf{x}| + \mathbf{y}^T\mathbf{y}\right)^{\frac{1}{2}} \leq \left(\mathbf{x}^T\mathbf{x} + 2(\mathbf{x}^T\mathbf{x})^{\frac{1}{2}}(\mathbf{y}^T\mathbf{y})^{\frac{1}{2}} + \mathbf{y}^T\mathbf{y}\right)^{\frac{1}{2}} \\ &= ||\mathbf{x}||_2 + ||\mathbf{y}||_2 \end{aligned}$$

1.5.1 Minkowski's inequality - triangle inequality for the p norm

$$||\mathbf{x}+\mathbf{y}||_p \leq ||\mathbf{x}||_p + ||\mathbf{y}||_p \tag{1.17}$$

Proof:

$$\sum |x_k + y_k|^p \leq \sum |x_k + y_k|^{p-1}|x_k| + \sum |x_k + y_k|^{p-1}|y_k| \tag{1.18}$$

Then by Hölder's inequality

$$\begin{aligned} \sum |x_k + y_k|^p &\leq \left(\sum |x_k + y_k|^{(p-1)q}\right)^{\frac{1}{q}} \left(\left(\sum |x_k|^p\right)^{\frac{1}{p}} + \left(\sum |y_k|^p\right)^{\frac{1}{p}}\right) \\ &= \left(\sum |x_k + y_k|^p\right)^{\frac{1}{q}} \left(\left(\sum |x_k|^p\right)^{\frac{1}{p}} + \left(\sum |y_k|^p\right)^{\frac{1}{p}}\right) \end{aligned} \tag{1.19}$$

since $p - 1 = \frac{p}{q}$. Divide both sides by

$$\left(\sum |x_k + y_k|^p\right)^{\frac{1}{q}} \tag{1.20}$$

to obtain the triangle inequality.

1.5.2 Equivalence of Norms for Finite Dimensional Vectors

For finite dimensional vectors all norms are equivalent in the sense that for 2 norms $M(\mathbf{x})$ and $N(\mathbf{x})$ there exist constants c_1, c_2 such that for all $\mathbf{x}$

$$c_1 M(\mathbf{x}) \leq N(\mathbf{x}) \leq c_2 M(\mathbf{x}) \tag{1.21}$$

$$\frac{1}{c_2} N(\mathbf{x}) \leq M(\mathbf{x}) \leq \frac{1}{c_1} N(\mathbf{x}) \tag{1.22}$$

This need only be proved for $M(\mathbf{x}) = N_\infty(\mathbf{x})$ since if M and N are both equivalent to N_∞ then they are also equivalent to each other. Suppose that

$$c_1 N_\infty \leq M \leq c_2 N_\infty \tag{1.23}$$

and

$$d_1 N_\infty \leq N \leq d_2 N_\infty \tag{1.24}$$

Then

$$c_1 N \leq d_2 M \tag{1.25}$$

and

$$d_1 M \leq c_2 N \tag{1.26}$$

where

$$\frac{c_1}{d_2} N \leq M \leq \frac{c_2}{d_1} N \tag{1.27}$$

Consider the unit cell S of vectors for which $N_s(\mathbf{x}) = 1$. Let $\mathbf{x}_0$ and $\mathbf{x}_1$ be elements such that

$$N(\mathbf{x}_0) = \min_{\mathbf{x}\in S} N(\mathbf{x}),\ \ N(\mathbf{x}_1) = \max_{\mathbf{x}\in S} N(\mathbf{x}) \tag{1.28}$$

(These exist since $N(\mathbf{x})$ is a continuous function of the elements of $\mathbf{x}$.)

Now for any $\mathbf{y}$, $\dfrac{\mathbf{y}}{N_\infty(\mathbf{y})}$ is in S, so

$$N(\mathbf{x}_0) \leq N\left(\frac{\mathbf{y}}{N_\infty(\mathbf{y})}\right) \leq N(\mathbf{x}_1) \tag{1.29}$$

or

$$N(\mathbf{x}_0) N_\infty(\mathbf{y}) \leq N(\mathbf{y}) \leq N(\mathbf{x}_1) N_\infty(\mathbf{y}) \tag{1.30}$$

This is the desired result with

$$c_1 = N(\mathbf{x}_0),\ \ c_2 = N(\mathbf{x}_1) \tag{1.31}$$

1.6 Matrix Norms

The induced norm of a matrix is defined as $||A|| = \sup\limits_{\mathbf{x}\neq 0} \dfrac{||A\mathbf{x}||}{||\mathbf{x}||}$.

$||A\mathbf{x}||$ is a continuous function of $||\mathbf{x}||$ so with $||\mathbf{x}|| = 1$

$$||A|| = \sup ||A\mathbf{x}|| \tag{1.32}$$

where the max is attained.

Let $\mathbf{x}$ be such that with $||\mathbf{x}|| = 1$

$$||A + B|| = ||(A + B)\mathbf{x}|| \leq ||A\mathbf{x}|| + ||B\mathbf{x}|| \leq ||A|| + ||B|| \tag{1.33}$$

so triangle inequality is satisfied. Also note that

$$||AB\mathbf{x}|| \leq ||A||\ ||B\mathbf{x}|| \leq ||A||\ ||B||\ ||\mathbf{x}|| \tag{1.34}$$

so that

$$||AB|| \leq ||A||\ ||B|| \tag{1.35}$$

1.6.1 Infinity Norm

Corresponding to $||\mathbf{x}||_\infty$ we have

$$||A||_\infty = \max_i \sum_j |a_{ij}| \text{ (max absolute row sum)} \tag{1.36}$$

since

$$||A\mathbf{x}||_\infty = \max_i \left| \sum_j a_{ij} x_j \right| \leq \max_i \left| \sum_j a_{ij} \right| \max_j |x_j| \leq \max_i \left| \sum_j a_{ij} \right| ||\mathbf{x}||_\infty \tag{1.37}$$

where the max is attained with $x_j = \text{sgn}(a_{ij})$.

1.6.2 1 Norm

Corresponding to $||\mathbf{x}||_1$ we have

$$||A||_1 = \max_j \sum_i |a_{ij}| \text{ (max absolute column sum)} \tag{1.38}$$

since

$$\begin{aligned} ||A\mathbf{x}||_1 = \sum_i \left| \sum_j a_{ij} x_j \right| &\leq \sum_i \sum_j |a_{ij}||x_j| = \sum_j \left(\sum_i |a_{ij}| \right) |x_j| \\ &\leq \max_j \left(\sum_i |a_{ij}| \right) \sum_j |x_j| = \max_j \left(\sum_i |a_{ij}| \right) ||\mathbf{x}||_1 \end{aligned} \tag{1.39}$$

with the max attained when $\mathbf{x} = e_j$, where j is the index of the max sum.

1.6.3 2 Norm

Corresponding to $||\mathbf{x}||_2$ we have

$$\frac{||A\mathbf{x}||_2^2}{||\mathbf{x}||_2^2} = \frac{\mathbf{x}^H A^H A \mathbf{x}}{\mathbf{x}^H \mathbf{x}} \tag{1.40}$$

where A^H is the Hermitian transpose of A.

Let u_i be an eigenvector of $A^H A$ with real eigenvalue σ^2. Set $\mathbf{x} = \sum \alpha_i u_i$, then

$$\frac{\mathbf{x}^H A^H A \mathbf{x}}{\mathbf{x}^H \mathbf{x}} = \frac{\sum \alpha_i^2 \sigma_i^2}{\sum \alpha_i^2} \leq \sigma_1^2 \tag{1.41}$$

where σ_1^2 is the largest eigenvalue of $A^H A$. The decomposition is possible because u_i are independent.

1.7 Norms of Functions

1.7.1 Function Space

For considering errors we need to measure differences between functions. It is convenient to regard functions as vectors in an infinite dimensional space. It turns out that many of the familiar concepts of three dimensional Euclidean geometry carry over to function spaces. In particular we can introduce norms to measure the size of a function, and an abstract definition of a generalized inner product of two functions, corresponding to the scalar product of two vectors. This leads to the concept of orthogonal functions. The introduction of axiomatic definition leads to proofs which hold for a variety of different norms and inner products. In order to enable standard arguments of real analysis to be carried over, function spaces are required to satisfy an axiom of completeness that the limit of a sequence of elements in a given space is contained in the space. Spaces with an inner product are known as Hilbert spaces in honor of their inventor.

1.7.2 Norms

Norms will be a measure of the size of a function, regarded as a vector. For example

$$||f|| = \left(\int_0^1 f^2 dx\right)^{\frac{1}{2}} \tag{1.42}$$

is a generalization of

$$||f|| = \left(\sum_1^3 f_i^2 dx\right)^{\frac{1}{2}} \tag{1.43}$$

for Euclidean space.

To be more precise we require the norm of a function to satisfy the following axioms:

1. $||f|| \geq 0$
2. $||\alpha f|| = |\alpha| ||f||$ for scalar α
3. $||f + g|| \leq ||f|| + ||g||$
4. $||f|| = 0$ if and only if $f = 0$

If a function is given by a table

$$f_i = f(x_i) \text{ for } i = 0, 1, 2, ..., n \tag{1.44}$$

it may be convenient to use as a measure

$$||f|| = \left(\sum_0^n f_i^2 dx\right)^{\frac{1}{2}} \tag{1.45}$$

Then we can have

$$||f|| = 0 \text{ for } f \neq 0 \tag{1.46}$$

This measure satisfies the first 3 axioms only. Such a measure is called a **semi-norm**.

Examples of norms are:

$$||f||_\infty = \max_{x\in[a,b]} |f(x)| \text{ Maximum norm} \tag{1.47}$$

$$||f||_2 = \left(\int_a^b f^2(x) dx\right)^{\frac{1}{2}} \text{ Euclidean norm} \tag{1.48}$$

$$||f||_1 = \int_a^b |f(x)|dx \tag{1.49}$$

These are special cases of

$$||f||_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \quad \text{for} \quad p \geq 1 \tag{1.50}$$

Note that, for example, we can regard $||f||_2$ as the limit of

$$||f|| = \left(\frac{1}{n}\sum_0^n f_i^2 dx\right)^{\frac{1}{2}} \tag{1.51}$$

where f_i is the table of values representing $f(x)$.

We can also introduce weighted norms such as the weighted Euclidean norm

$$||f|| = \left(\int_a^b f^2(x)w(x)dx\right)^{\frac{1}{2}} \tag{1.52}$$

where $w(x)$ is a non-negative weight function.

All these norms can be shown to satisfy the axioms.

1.7.3 Non-equivalence of Norms for Functions

For functions norms are not equivalent. If $p > q$ then

$$||f||_p \to 0 \text{ implies } ||f||_q \to 0 \tag{1.53}$$

but not the other way round.

For example consider

$$f_n = \begin{cases} 1 & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases} \tag{1.54}$$

Then

$$\int_{-1}^1 f_n^2 dx = \frac{1}{n} \tag{1.55}$$

and

$$||f_n||_2 = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty \tag{1.56}$$

while

$$||f_n||_\infty = 1 \tag{1.57}$$

On the other hand if

$$||f||_\infty = \epsilon \tag{1.58}$$

then

$$\int_a^b f^2 dx \leq \epsilon^2(b-a) \tag{1.59}$$

so

$$||f||_2 \leq \epsilon\sqrt{b-a} \;\to 0 \;\text{ as }\; \epsilon \to 0 \tag{1.60}$$

1.8 Inner Products of Functions: Orthogonality

1.8.1 Orthogonality

It is also convenient to introduce the idea of orthogonality in function space. Define the inner product as

$$\begin{aligned}(f, g) &= \int_a^b f(x)\ g(x)\ w(x)\ dx \ \text{ in the continuous case} \\ &= \sum_{i=0}^{n} f_i\ g_i\ w_i \ \text{ in the discrete case}\end{aligned} \tag{1.61}$$

Then if (f, g) vanishes the functions are said to be orthogonal.

We have

$$\begin{aligned}(f, g) = (g, f) & \qquad \text{commutative} \\ (c_1 f + c_2 g, \Phi) = c_1(f, \Phi) + c_2(g, \Phi) & \qquad \text{linearity} \\ (f, f) \geq 0 & \qquad \text{positivity}\end{aligned} \tag{1.62}$$

These can be taken as axioms for an abstract inner product.

1.8.2 Triangle Inequality for the ∞ and 1 Norms

We have

$$\max|f + g| \leq \max(|f| + |g|) \leq \max|f| + \max|g| \tag{1.63}$$

and

$$\int |f + g| dx \leq \int (|f| + |g|) dx \tag{1.64}$$

Thus the ∞ and 1 norms satisfy the triangle inequality. They also already satisfy axioms (1), (2) and (4).

1.8.3 Cauchy Schwarz Inequality

$$(f, g)^2 \leq (f, f)(g, g) \tag{1.65}$$

Proof:

$$0 \leq (f + pg, f + pg) \leq (f, f) + 2p|(f, g)| + p^2(g, g) \tag{1.66}$$

But since this is a quadratic function of p it has imaginary roots or coincident real roots. The discriminant yields the desired inequality.

Or set

$$p = -\frac{|(f, g)|}{(g, g)} \tag{1.67}$$

and multiply by (g, g) to get

$$0 \leq (f, f)\ (g, g) - 2|(f, g)|^2 + |(f, g)|^2 \tag{1.68}$$

1.8.4 Triangle Inequality for the Euclidean Norm

$$\begin{aligned} ||f+g||_2 &= (f+g, f+g)^{\frac{1}{2}} \\ &\le ((f,f) + 2|(f,g)| + (g,g))^{\frac{1}{2}} \\ &\le \Big((f,f) + 2(f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}} + (g,g)\Big)^{\frac{1}{2}} \\ &= ||f|| + ||g|| \end{aligned} \tag{1.69}$$

using the Schwartz inequality.

1.8.5 Pythagorean Theorem

In the Euclidean norm, if $(f,g) = 0$ then

$$||f+g||^2 = ||f||^2 + ||g||^2 \tag{1.70}$$

Proof:

$$||f+g||^2 = (f+g, f+g) = (f,f) + (f,g) + (g,f) + (g,g) \tag{1.71}$$

1.8.6 Linear Independence

A set of functions $\phi_0, \phi_1, ..., \phi_n$ is said to be linearly independent if

$$\left|\left|\sum c_i \ \phi_i\right|\right| = 0 \tag{1.72}$$

implies

$$c_i = 0 \text{ for } i = 0, 1, ..., n \tag{1.73}$$

Orthogonal functions are independent since then from the Pythagorean theorem

$$\left|\left|\sum_{i=0}^{n} c_i \ \phi_i\right|\right|^2 = \sum_{i=0}^{n} c_i^2 ||\phi_i||^2 \tag{1.74}$$

1.8.7 Weierstrass Theorem

Let $f(x)$ be given in interval $[a,b]$. Let the lower bound of the error in the maximum norm for all polynomials of order n be

$$E_n(f) = \min_{p_n(x)} ||f - p_n(x)||_\infty \tag{1.75}$$

Then if f is continuous

$$\lim_{n\to\infty} E_n(f) = 0 \tag{1.76}$$

That is, a continuous function can be arbitrarily well approximated in a closed interval by a polynomial of sufficiently high order.

The proof is by construction of the required $p_n(x)$ (Isaacson and Keller, p183).

1.9 Portraits of Courant and Hilbert

Figure 1.1: Richard Courant (1888-1972)

Figure 1.2: David Hilbert (1862-1943)