Advanced Computational Fluid Dynamics
AA215A Lecture 2
Approximation Theory

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Contents

Lecture 2

Approximation Theory

2.1 Least Squares Approximation and Projection

Suppose that we wish to approximate a given function f by a linear combination of independent basis functions $\phi_j, j = 0, .., n$. In general we wish to choose coefficients c_j to minimize the error

$$||f - \sum_{j=0}^{n} c_j \phi_j|| \tag{2.1}$$

in some norm. Ideally we might use the infinity norm, but it turns out it is computationally very expensive to do this, requiring an iterative process. The best known method for finding the best approximation in the infinity norm is the exchange algorithm (M.J.D.Powell) On the other hand it is very easy to calculate the best approximation in the Euclidean norm as follows. Choose c_j to minimize the least squares integral

$$J(c_0, c_1, c_2, ...c_n) = \int \left(\sum_{j=0}^{n} c_j \phi_j - f \right)^2 dx \tag{2.2}$$

equivalent to minimizing the Euclidean norm. Then we require

$$0 = \frac{\partial J}{\partial c_i} = \int 2 \left(\sum_{j=0}^{n} c_j \phi_j - f \right) \phi_i dx \tag{2.3}$$

Thus the best approximation is

$$f^* = \sum_{j=0}^{n} c_j^* \phi_j \tag{2.4}$$

Where

$$\left(\sum_{j=0}^{n} c_j^* \phi_j - f, \phi_i \right) = (f^* - f, \phi_i) = 0, \text{ for } i = 0, 1, 2, ..., n \tag{2.5}$$

or

$$\sum_{j=0}^{n} a_{ij} c_j^* = b_i \tag{2.6}$$

where

$$a_{ij} = (\phi_i, \phi_j),\ \ b_i = (f, \phi_i) \tag{2.7}$$

Notice that since f^* is a linear combination of the ϕ_j these equations state that

$$(f - f^*, f^*) = 0 \tag{2.8}$$

Thus the least squares approximation f^* is orthogonal to the error $f - f^*$. This means that f^* is the projection of f onto the space spanned by the basis functions ϕ_j, in the same way that a three dimensional vector might be projected onto a plane.

If the ϕ_j form an orthonormal set,

$$a_{ij} = \delta_{ij} \tag{2.9}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, and we have directly

$$c_i^* = (f, \phi_i) \tag{2.10}$$

2.2 Alternative Proof of Least Squares Solution

Let

$$f^* = \sum c_j^* \phi_j \tag{2.11}$$

be the best approximation and consider another approximation

$$\sum c_j \phi_j \tag{2.12}$$

Then if $f - f^*$ is orthogonal to all ϕ_j

$$\begin{aligned} \left|\left|\sum c_j \phi_j - f\right|\right|^2 &= \left|\left|\sum (c_j - c_j^*)\phi_j - (f - f^*)\right|\right|^2 && (2.13) \\ &= \left|\left|\sum (c_j - c_j^*)\phi_j\right|\right|^2 + ||(f - f^*)||^2 && (2.14) \\ &\geq ||(f - f^*)||^2 && (2.15) \end{aligned}$$

Recall that in the Lecture Notes $|| \cdot || = || \cdot ||_2 = (\cdot, \cdot)^{1/2}$

2.3 Bessel's Inequality

Since f^* is orthogonal to $f - f^*$ the Pythagorean theorem gives

$$||f||^2 = ||f^* + f - f^*||^2 = ||f^*||^2 + ||f - f^*||^2 \tag{2.16}$$

Also if the ϕ_j are orthonormal,

$$||f^*||^2 = \sum c_j^{*2} \tag{2.17}$$

so we have Bessel's inequality

$$\sum c_j^{*2} \leq ||f||^2 \tag{2.18}$$

Thus the series $\sum c_j^2$ is convergent and if $||f^* - f|| \to 0$ as $n \to \infty$ we have Parseval's formula

$$\sum c_j^{*2} = ||f||^2 \tag{2.19}$$

This is the case for orthogonal polynomials because of the Weiesstrass theorem.

2.4 Orthonormal Systems

The functions ϕ_i form an orthonormal set if they are orthogonal and $||\phi_i|| = 1$. We can write

$$(\phi_i, \phi_j) = \delta_{ij} \tag{2.20}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.

An example of an orthogonal set is

$$\phi_j = \cos(jx), \text{ for } j = 0, 1, ..., n \tag{2.21}$$

on the ineral $[0, \pi]$. Then if $j \neq k$

$$(\phi_j, \phi_k) = \int_0^\pi \cos(jx)\cos(kx)\ dx = \int_0^\pi \frac{1}{2}\left(\cos(j-k)x + \cos(j+k)x\right)\ dx = 0 \tag{2.22}$$

Also

$$(\phi_j, \phi_j) = \int_0^\pi \cos^2(jx)dx = \int_0^\pi \frac{1}{2}\left(1 + \cos(2jx)\right)\ dx = \frac{\pi}{2} \tag{2.23}$$

2.5 Construction of Orthogonal Functions

This can be done by Gram Schmidt orthogonalization. Define the inner product

$$(f, g) = \int_a^b f(x)g(x)dx \tag{2.24}$$

Functions g_i are called independent if

$$\sum_{i=0}^{n} \alpha_i g_i = 0 \text{ implies } \alpha_i = 0 \text{ for } i = 0, ..., n \tag{2.25}$$

Let $g_i(x), i = 0, 1, ..., n$, be any set of independent functions. Then set

$$f_0(x) = d_0 g_0(x) \tag{2.26}$$

$$f_1(x) = d_1\left(g_1(x) - c_{01} f_0(x)\right) \tag{2.27}$$

$$\vdots$$

$$f_n(x) = d_n\left(g_n(x) - c_{0n} f_0(x) ... - c_{n-1,n} f_{n-1}(x)\right) \tag{2.28}$$

We require

$$(f_i, f_j) = \delta_{ij} \tag{2.29}$$

This gives

$$d_0 = \frac{1}{\sqrt{(g_0, g_0)}} \tag{2.30}$$

Then $0 = (f_0, f_1) = d_1\left((f_0, g_1) - c_{01}\right)$ and in general

$$c_{jk} = (f_j, g_k) \tag{2.31}$$

while the d_k are easily obtained from

$$(f_k, f_k) = 1 \tag{2.32}$$

2.6 Orthogonal Polynomials

A similar approach can be used to construct orthogonal polynomials. Given orthogonal polynomials, $\phi_j(x),\ j = 0, 1, \ldots, n$, we can generate a polynomial of degree $n+1$ by multiplying $\phi_n(x)$ by x, and we can make it orthogonal to the previous $\phi_j(x)$ by subtracting a linear combination of these polynomials. Thus, we set

$$\phi_{n+1}(x) = \alpha_n x\phi_n(x) - \sum_{i=0}^{n} c_{ni}\phi_i(x) \tag{2.33}$$

Now,

$$\alpha_n(x\phi_n, \phi_j) - \sum_{i=0}^{n} c_{ni}(\phi_i, \phi_j) = 0 \tag{2.34}$$

But $(\phi_i, \phi_j) = 0$ for $i \neq j$ so

$$c_{nj}||\phi_j||^2 = \alpha_n(x\phi_n, \phi_j) = \alpha_n(\phi_n, x\phi_j) \tag{2.35}$$

Since $x\phi_j$ is a polynomial of order $j+1$ this vanishes except for $j = n-1, n$.

Thus

$$\phi_{n+1}(x) = \alpha x\phi_n(x) - c_{nn}\phi_n(x) - c_{n,n-1}\phi_{n-1}(x) \tag{2.36}$$

where

$$c_{nn} = \frac{\alpha_n(\phi_n, x\phi_n)}{||\phi_n||^2}\ ,\ c_{n,n-1} = \frac{\alpha_n(\phi_n, x\phi_{n-1})}{||\phi_{n-1}||^2} \tag{2.37}$$

and α is arbitrary.

2.7 Zeros of Orthogonal Polynomials

An orthogonal polynomial $P_n(x)$ for the interval $[a, b]$ has n distinct zeros in $[a, b]$. Suppose it had only $k < n$ zeros $x_1, x_2, ..., x_k$ at which $P_n(x)$ changes sign. Then consider

$$Q_k(x) = (x - x_1) \ldots (x - x_k) \tag{2.38}$$

where for $k = 0$ we define $Q_0(x) = 1$. Then $Q_k(x)$ changes sign at the same points as $P_n(x)$, and

$$\int_a^b P_n(x)Q_k(x)w(x)dx \neq 0 \tag{2.39}$$

But this is impossible since $P_n(x)$ is orthogonal to all polynomials of degree $< n$.

2.8 Legendre Polynomials

These are orthogonal in the interval $[-1, 1]$ with a constant weight function $w(x) = 1$. They can be generated from Rodrigues' formula

$$L_0(x) = 1 \tag{2.40}$$

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2 - 1)^n\right],\ \text{for } n = 1, 2, ... \tag{2.41}$$

The first few are

$$\begin{aligned} L_0 &= 1 \\ L_1(x) &= x \\ L_2(x) &= \frac{1}{2}(3x^2 - 1) \\ L_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ L_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\ L_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned} \tag{2.42}$$

where they can be successively generated for $n \geq 2$ by the recurrence formula

$$L_{n+1} = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x) \tag{2.43}$$

They are alternately odd and even and are normalized so that

$$L_n(1) = 1\ ,\ L_n(-1) = (-1)^n \tag{2.44}$$

while for $|x| < 1$

$$|L_n(x)| < 1 \tag{2.45}$$

Also

$$(L_n, L_j) = \int_{-1}^{1} L_n(x) L_j(x) dx = \begin{cases} 0 & \text{if } n \neq j \\ \frac{2}{2n+1} & \text{if } n = j \end{cases} \tag{2.46}$$

Writing the recurrence relation as

$$L_n(x) = \frac{2n-1}{n} x L_{n-1}(x) - \frac{n-1}{n} L_{n-2}(x) \tag{2.47}$$

it can be seen that the leading coefficient of $L_n(x)$ is multiplied by $(2n-1)/n$ when $L_n(x)$ is generated. Hence the leading coefficient is

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{1 \cdot 2 \cdot 3 \cdot ... \cdot n} = \frac{1}{2^n} \frac{2n!}{(n!)^2} \tag{2.48}$$

$L_n(x)$ also satisfies the differential equation

$$\frac{d}{dx}\left[\left(1 - x^2\right) \frac{d}{dx} L_n(x)\right] + n(n+1) L_n(x) = 0 \tag{2.49}$$

2.9 Chebyshev Polynomials

Let

$$x = \cos\ \phi,\ \ 0 \leq \phi \leq \pi \tag{2.50}$$

and define

$$T_n(x) = \cos(n\phi) \tag{2.51}$$

in the interval [-1,1]. Since

$$\cos(n+1)\phi + \cos(n-1)\phi = 2\cos\phi\cos n\phi,\ \ n \geq 1 \tag{2.52}$$

we have

$$T_0(x) = 1 \tag{2.53}$$
$$T_1(x) = x \tag{2.54}$$
$$\vdots$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{2.55}$$

These are the Chebyshev polynomials.
Property 1:
Leading coefficient is 2^{n-1} for $n \geq 1$
Property 2:

$$T_n(-x) = (-1)^n T_n(x) \tag{2.56}$$

Property 3:
$T_n(x)$ has n zeros at

$$x_k = \cos\left(\frac{2k-1}{n}\frac{\pi}{2}\right) \text{ for } k = 1, ..., n \tag{2.57}$$

and $n+1$ extrema in $[-1,1]$ with the values $(-1)^k$, at

$$x'_k = \cos\frac{k\pi}{n} \text{ for } k = 0, ..., n \tag{2.58}$$

Property 4 (Continuous orthogonality):
They are orthogonal in the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}} \tag{2.59}$$

since

$$(T_r, T_s) = \int_0^\pi \cos r\theta \cos s\theta \ d\theta = \begin{cases} 0 & \text{if } r \neq s \\ \frac{\pi}{2} & \text{if } r = s \neq 0 \\ \pi & \text{if } r = s = 0 \end{cases} \tag{2.60}$$

Property 5 (Discrete orthogonality):
They are orthogonal in the discrete inner product

$$(f,g) = \sum_{j=0}^{n} f(x_j)g(x_j) \tag{2.61}$$

where x_j are the zeros of $T_{n+1}(x)$ and

$$\arccos x_j = \theta_j = \frac{2j+1}{2n+1}\frac{\pi}{2},\ \ j = 0, 1, ..., n \tag{2.62}$$

Then

$$(T_r, T_s) = \sum_{j=0}^{n} \cos r\theta_j \cos s\theta_j = \begin{cases} 0 & \text{if } r \neq s \\ \frac{n+1}{2} & \text{if } r = s \neq 0 \\ n+1 & \text{if } r = s = 0 \end{cases} \tag{2.63}$$

Property 6 (Minimax):
$\left(1/2^{n-1}\right) T_n(x)$ has smallest maximum norm in $[-1,1]$ of all polynomials with leading coefficient unity.
Proof:
Suppose $||p_n(x)||_\infty$ were smaller. Now at the $n+1$ extrema $x_0, ..., x_n$ of $T_n(x)$, which all have the same magnitude of unity,

$$p_n(x_n) < \frac{1}{2^{n-1}} T_n(x_n) \tag{2.64}$$

$$p_n(x_{n-1}) > \frac{1}{2^{n-1}} T_n(x_{n-1}) \tag{2.65}$$

$$\vdots$$

Thus $p_n(x) - \left(1/2^{n-1}\right) T_n(x)$ changes sign n times in $[-1,1]$, but this is impossible since it is of degree $n-1$ and has $n-1$ roots.

2.10 Best Approximation Polynomial

Let $f(x) - P_n(x)$ have maximum deviations, $+e, -e$, alternately at $n+2$ points $x_0, ..., x_{n+1}$ in $[a, b]$. Then $P_n(x)$ minimizes $||f(x) - P_n(x)||_\infty$
Proof: Suppose $||f(x) - Q_n(x)||_\infty < |e|$, then we have at $x_0, ..., x_{n+1}$

$$Q_n(x) - P_n(x) = f(x) - P_n(x) - (f(x) - Q_n(x)) \tag{2.66}$$

has same sign as $f(x) - P_n(x)$. Thus it has opposite sign at $n+2$ points, giving $n+1$ sign changes. But this is impossible since it has only n roots.

2.11 Best Interpolating Polynomial

The remainder for the Lagrange interpolation polynomial is

$$f(x) - P_n(x) = w_n(x) \frac{f^{n+1}(\xi)}{(n+1)!} \tag{2.67}$$

where

$$w_n(x) = (x - x_0)(x - x_1) \dots (x - x_n) \tag{2.68}$$

Thus we can minimize $||w_n(x)||_\infty$ in $[-1,1]$ by making x_k the zeros of $T_{n+1}(x)$. Then

$$w_n(x) = \left(\frac{1}{2}\right)^n T_{n+1}(x) \tag{2.69}$$

$$||w_n(x)||_\infty = \left(\frac{1}{2}\right)^n \tag{2.70}$$

2.12 Jacobi Polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal in the interval $[-1,1]$ for the weight function

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta} \tag{2.71}$$

Thus the family of Jacobi polynomials includes both the Legendre polynomials

$$L_n(x) = P_n^{(0,0)}(x) \tag{2.72}$$

and the Chebyshev polynomials

$$T_n(x) = P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{2.73}$$

2.13 Fourier Series

Let $f(\theta)$ be periodic with period 2π and square integrable on $[0,2\pi]$. Consider the approximation of $f(\theta)$

$$S_n(\theta) = \frac{1}{2}a_0 + \sum_{r=1}^{n}(a_r \cos r\theta + b_r \sin r\theta) \tag{2.74}$$

Let us minimize

$$||f - S_n||_2 = \left\{ \int_0^{2\pi} \big(f(\theta) - S_n(\theta)\big)^2 d\theta \right\}^{\frac{1}{2}} \tag{2.75}$$

Let

$$J(a_0, a_1, ..., a_n, b_1, ..., b_n) = ||f - S_n||^2 \tag{2.76}$$

The trigonometric functions satisfy the orthogonality relations

$$\int_0^{2\pi} \cos(r\theta)\cos(s\theta)d\theta = \begin{cases} 0 & \text{if } r \neq s \\ \pi & \text{if } r = s \neq 0 \end{cases} \tag{2.77}$$

$$\int_0^{2\pi} \sin(r\theta)\sin(s\theta)d\theta = \begin{cases} 0 & \text{if } r \neq s \\ \pi & \text{if } r = s \neq 0 \end{cases} \tag{2.78}$$

$$\int_0^{2\pi} \sin(r\theta)\cos(s\theta)d\theta = 0 \tag{2.79}$$

At the minimum allowing for these

$$0 = \frac{\partial J}{\partial a_r} = -2\int_0^{2\pi} f(\theta)\cos(r\theta)d\theta + 2a_r \int_0^{2\pi} \cos^2(r\theta)d\theta \tag{2.80}$$

whence

$$a_r = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(r\theta)d\theta \tag{2.81}$$

and similarly

$$b_r = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(r\theta)d\theta \tag{2.82}$$

Also since $||f - S_n||^2 \geq 0$ we obtain Bessel's inequality

$$\frac{1}{2}a_0^2 + \sum_{r=1}^{n} \left(a_r^2 + b_r^2\right) \leq \frac{1}{\pi}\int_0^{2\pi} f^2(\theta)d\theta \tag{2.83}$$

and since the right hand side is independent of n the sum converges and

$$\lim_{r\to\infty} |a_r| = 0, \quad \lim_{r\to\infty} |b_r| = 0, \tag{2.84}$$

Also

$$\lim_{n\to\infty} ||f - S_n||^2 = 0 \tag{2.85}$$

If not, it would violate Weierstrass's theorem translated to trigonometric functions (Isaacson and Keller, p 230, p 198).

2.14 Error Estimate for Fourier Series

Let $f(\theta)$ have K continuous derivatives. Then the Fourier coefficients decay as

$$|a_r| = \mathcal{O}\left(\frac{1}{r^K}\right) \ , \ |b_r| = \mathcal{O}\left(\frac{1}{r^K}\right) \tag{2.86}$$

and

$$|f(\theta) - S_n(\theta)| = \mathcal{O}\left(\frac{1}{n^{K-1}}\right) \tag{2.87}$$

Integrating repeatedly by parts

$$\begin{aligned} a_r &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(r\theta)d\theta \\ &= -\frac{1}{r\pi}\int_0^{2\pi} f'(\theta)\sin(r\theta)d\theta \\ &= \frac{1}{r^2\pi}\int_0^{2\pi} f''(\theta)\cos(r\theta)d\theta \\ &\vdots \end{aligned} \tag{2.88}$$

Let $M = \sup|f^{(K)}(\theta)|$. Then since $|\cos(r\theta)| \leq 1$ and $|\sin(r\theta)| \leq 1$

$$|a_r| \leq \frac{2M}{r^K} \tag{2.89}$$

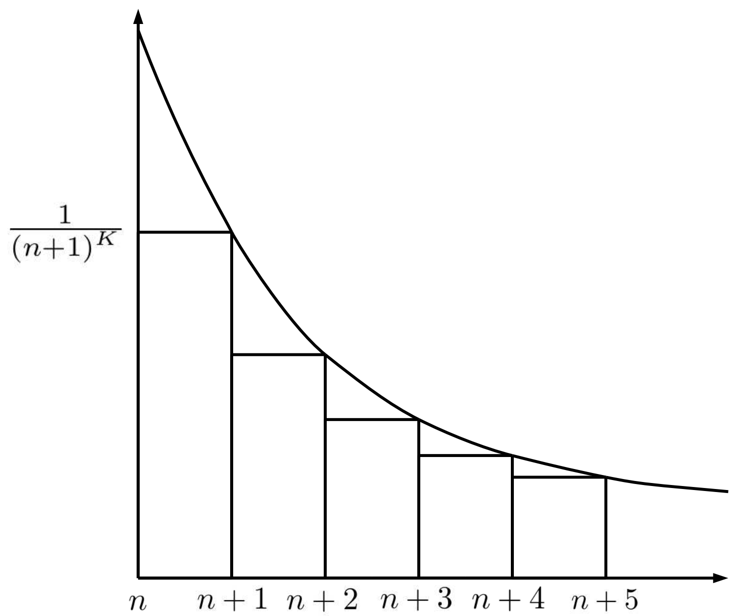
and similarly

$$|b_r| \leq \frac{2M}{r^K} \tag{2.90}$$

Also

$$
\begin{aligned}
|S_m(\theta) - S_n(\theta)| &\le \left| \sum_{r=n+1}^{m} (a_r \cos r\theta + b_r \sin r\theta) \right| \\
&\le \sum_{r=n+1}^{m} \frac{4M}{r^K} \\
&\le 4M \int_n^\infty \frac{d\xi}{\xi^K} \\
&= \frac{4M}{K-1} \frac{1}{n^{K-1}}
\end{aligned}
\tag{2.91}
$$

where the integral contains the sum of the rectangles shown in the sketch.



Thus the partial sums $S_n(\theta)$ form a Cauchy sequence and converge uniformly to $f(\theta)$, and

$$
|f(\theta) - S_n(\theta)| \le \frac{4M}{K-1} \frac{1}{n^{K-1}} \tag{2.92}
$$

2.15 Orthogonality Relations for Discrete Fourier Series

Consider

$$
S = \sum_{j=0}^{2N-1} e^{ir\theta_j} = \sum_{j=0}^{2N-1} \omega^{rj} \tag{2.93}
$$

where

$$
\omega = e^{i\frac{\pi}{N}},\ \omega^{2N} = 1 \tag{2.94}
$$

Then

$$
S = \frac{1 - \omega^{2Nr}}{1 - \omega^r} = \begin{cases} 0 & \text{if } \omega^r \neq 1 \\ 2N & \text{if } \omega^r = 1 \end{cases} \tag{2.95}
$$

Therefore

$$\sum_{j=0}^{2N-1} e^{ir\theta_j} e^{-is\theta_j} = \begin{cases} 0 & \text{if } |r-s| \neq 0, 2N, 4N \\ 2N & \text{if } |r-s| = 0, 2N, 4N \end{cases} \tag{2.96}$$

Taking the real part

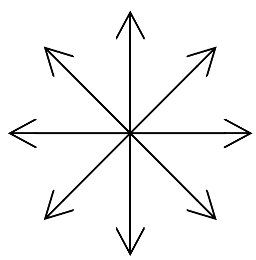
$$\sum_{j=0}^{2N-1} \cos(r-s)\theta_j = 0 \text{ if } |r-s| \neq 0, 2N, 4N \tag{2.97}$$

$$\sum_{j=0}^{2N-1} \cos(r+s)\theta_j = 0 \text{ if } |r+s| \neq 0, 2N, 4N \tag{2.98}$$

so

$$\sum_{j=0}^{2N-1} \cos r\theta_j \cos s\theta_j = \frac{1}{2} \sum_{j=0}^{2N-1} \big(\cos(r-s)\theta_j + \cos(r+s)\theta_j \big) = \begin{cases} 0 & \text{if } r \neq s \\ N & \text{if } r = s \neq 0, N \\ 2N & \text{if } r = s = 0, N \end{cases} \tag{2.99}$$

Note that $\sum_{j=0}^{2N-1} w^{rj}$ is the sum of $2N$ equally spaced unit vectors as sketched, except in the case when $\omega^r = 1$.



2.16 Trigonometric Interpolation

Instead of finding a least square fit we can calculate the coefficients of the sum so that the sum exactly fits the function at equal intervals around the circle. Let A_r and B_r be chosen so that

$$U_n(\theta) = \frac{1}{2}A_0 + \sum_{r=1}^{n-1}(A_r \cos r\theta + B_r \sin r\theta) + \frac{1}{2}A_n \cos n\theta = f(\theta) \tag{2.100}$$

at the $2n+1$ points

$$\theta_j = \frac{j}{2n}2\pi \tag{2.101}$$

It is easy to verify that

$$\sum_{j=0}^{2n-1} \cos r\theta_j \cos s\theta_j = \begin{cases} 0 & \text{if } r \neq s \\ n & \text{if } r = s \neq 0, n \\ 2n & \text{if } r = s = 0, n \end{cases} \tag{2.102}$$

$$\sum_{j=0}^{2n-1} \sin r\theta_j \sin s\theta_j = \begin{cases} 0 & \text{if } r \neq s \\ n & \text{if } r = s \neq 0, n \\ 0 & \text{if } r = s = 0, n \end{cases} \tag{2.103}$$

$$\sum_{j=0}^{2n-1} \sin r\theta_j \cos s\theta_j = 0 \tag{2.104}$$

It follows on multiplying through by $\cos r\theta_j$ or $\sin r\theta_j$ and summing over θ_j that

$$A_r = \frac{1}{n} \sum_{j=0}^{2n-1} f(\theta_j) \cos r\theta_j \tag{2.105}$$

$$B_r = \frac{1}{n} \sum_{j=0}^{2n-1} f(\theta_j) \sin r\theta_j \tag{2.106}$$

for $r = 1, ..., n$. Note that writing

$$c_0 = \frac{1}{2} A_0 \tag{2.107}$$

$$c_r = \frac{1}{2}(A_r - iB_r) \tag{2.108}$$

$$c_{-r} = \frac{1}{2}(A_r + iB_r) \tag{2.109}$$

the sum can be expressed as

$$U_n(\theta) = \sum_{r=-n}^{n-1} c_r e^{ir\theta} = {\sum_{r=-n}^{n}}' c_r e^{ir\theta} \tag{2.110}$$

where

$$c_r = \frac{1}{2n} \sum_{j=0}^{2n-1} f(\theta_j) e^{-ir\theta_j} \tag{2.111}$$

and $\sum'$ is defined as

$$ {\sum_{r=-n}^{n}}' a_r = \sum_{r=-n}^{n} a_r - \frac{1}{2}\left(a_n + a_{-n}\right) \tag{2.112}$$

2.17 Error Estimate for Trigonometric Interpolation

In order to estimate the error of the interpolation we compare the interpolation coefficients A_r and B_r with the Fourier coefficients a_r and b_r. The total error can then be estimated as the sum of the error in the truncated Fourier terms and the additional error introduced by replacing the Fourier coefficients by the interpolation coefficients. The interpolation coefficients can be expressed in terms of the Fourier coefficients by substituting the infinite Fourier series for $f(\theta)$ into the formula for the interpolation coefficients. For this purpose it is convenient to use the complex form. Thus

$$C_r = \frac{1}{2n} \sum_{j=0}^{2n-1} f(\theta_j) e^{-ir\theta_j} \tag{2.113}$$

$$= \frac{1}{2n} \sum_{j=0}^{2n-1} e^{-ir\theta_j} \sum_{k=-\infty}^{\infty} c_k e^{ik\theta_j} \tag{2.114}$$

$$= \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{k=-\infty}^{\infty} c_k e^{-i(r-k)\theta_j} \tag{2.115}$$

Here

$$\sum e^{i(r-k)\theta_j} = \begin{cases} 2n & \text{if } r-k = 2qn \text{ where } q = 0, 1, ... \\ 0 & \text{if } r-k \neq 2qn \end{cases} \tag{2.116}$$

Hence

$$C_r = c_r + \sum_{k=1}^{\infty}(c_{2kn+r} + c_{2kn-r}) \tag{2.117}$$

and similarly

$$C_{-r} = c_{-r} + \sum_{k=1}^{\infty}(c_{2kn+r} + c_{2kn-r}) \tag{2.118}$$

Accordingly

$$A_r = C_r + C_{-r} = a_r + \sum_{k=1}^{\infty}(a_{2kn+r} + a_{2knr-r}) \tag{2.119}$$

and similarly

$$B_r = i(C_r - C_{-r}) = b_r + \sum_{k=1}^{\infty}(b_{2kn+r} - b_{2kn-r}) \tag{2.120}$$

Thus the difference between the interpolation and Fourier coefficients can be seen to be an aliasing error in which all the higher harmonics are lumped into the base modes.

Now we can estimate $|A_r - a_r|$, $|B_r - b_r|$ using

$$|a_r| \leq \frac{2M}{r^K}, \quad |b_r| \leq \frac{2M}{r^K}. \tag{2.121}$$

We get

$$\begin{aligned} |A_r - a_r| &\leq \sum_{k=1}^{\infty} 2M \left| \frac{1}{(2kn-r)^K} + \frac{1}{(2kn+r)^K} \right| \\ &= \frac{2M}{(2n)^K} \sum_{k=1}^{\infty} \frac{1}{k^K} \left| \frac{1}{(1-\frac{r}{2kn})^K} + \frac{1}{(1+\frac{r}{2kn})^K} \right| \end{aligned} \tag{2.122}$$

Since $r \leq n$ this is bounded by

$$\frac{2M}{(2n)^K} \sum_{k=1}^{\infty} \frac{1}{k^K}(2^k + 1) \leq \frac{2(2^K+1)M}{2^K n^K} \left(1 + \int_1^{\infty} \frac{d\xi}{\xi^K}\right) \leq \frac{2(2^K+1)M}{2^K n^K} \left(1 + \frac{1}{K-1}\right) \tag{2.123}$$

Using a similar estimate for B_r we get

$$|A_r - a_r| < \frac{5M}{n^K}, \quad |B_r - b_r| < \frac{5M}{n^K} \tag{2.124}$$

for $K \geq 2$. Finally we can estimate the error of the interpolation sum $u_n(\theta)$ as

$$|f(\theta) - u_n(\theta)| \leq |f(\theta) - S_n(\theta)| + |S_n(\theta) - u_n(\theta)| \tag{2.125}$$

$$< \frac{4M}{K-1}\frac{1}{n^{K-1}} + \sum_{r=0}^{n}\left\{|A_r - a_r| + |B_r - b_r|\right\} \tag{2.126}$$

$$< \frac{4M}{K-1}\frac{1}{n^{K-1}} + \frac{10M(n+1)}{n^K} \tag{2.127}$$

2.18 Fourier Cosine Series

For a function defined for $0 \leq \theta \leq \pi$ the Fourier cosine series is

$$S_c(\theta) = \sum_{r=0}^{\infty} a_r \cos r\theta \tag{2.128}$$

where

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(\theta) d\theta \tag{2.129}$$

$$a_r = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos(r\theta) d\theta, r > 0 \tag{2.130}$$

Hence since $\frac{d}{d\theta} \cos(r\theta) = 0$ at $\theta = 0, \pi$ the function will not be well approximated at $0, \pi$ unless $f'(\theta) = f'(\pi) = 0$. The cosine series in fact implies an even continuation at $\theta = 0, \pi$.

Now if all derivatives of $f(\theta)$ of order $0, 1, ..., K-1$, are continuous, and $f^{(p)}(0) = f^{(p)}(\pi) = 0$ for all odd $p < K$ and $f^{(K)}(\theta)$ is integrable, then integration by parts from the coefficients of $f^{(K)}(\theta)$ gives

$$|a_r| \leq \frac{M}{r^k} \tag{2.131}$$

2.19 Cosine Interpolation

Let $f(\theta)$ be approximated by

$$U_n(\theta) = \sum_0^n a_r \cos r\theta \tag{2.132}$$

and let

$$U_n(\theta_j) = f(\theta_j) \text{ for } j = 0, 1, ...n \tag{2.133}$$

where

$$\theta_j = \frac{2j+1}{n+1} \frac{\pi}{2} \tag{2.134}$$

Then

$$\sum_{j=0}^{n} \cos r\theta_j \cos s\theta_j = \begin{cases} 0 & \text{if } 0 \leq r \neq s \leq n \\ \frac{n+1}{2} & \text{if } 0 \leq r = s \leq n \\ n+1 & \text{if } 0 = r = s \end{cases} \tag{2.135}$$

Now if we multiply the first equation by $\cos s\theta_j$ and sum, we find that

$$a_r = \frac{2}{n+1} \sum_{j=0}^{n} f(\theta_j) \cos r\theta_j \text{ for } 0 < r < n \tag{2.136}$$

$$a_0 = \frac{1}{n+1} \sum_{j=0}^{n} f(\theta_j) \tag{2.137}$$

2.20 Chebyshev Expansions

Let

$$g(x) = \sum_{r=0}^{\infty} a_r T_r(x) \tag{2.138}$$

approximate $f(x)$ in the interval $[-1, 1]$. Then $G(\theta) = g(\cos\theta)$ is the Fourier cosine series for $F(\theta) = f(\cos\theta)$ for $0 \le \theta \le \pi$, since

$$T_r(\cos\theta) = \cos r\theta \tag{2.139}$$

$$G(\theta) = g(\cos\theta) = \sum_{r=0}^{\infty} a_r \cos r\theta \tag{2.140}$$

Thus

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta = \frac{1}{\pi}\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}} \tag{2.141}$$

and

$$a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta d\theta = \frac{2}{\pi}\int_{-1}^{1} f(x)T_r(x)\frac{dx}{\sqrt{1-x^2}} \tag{2.142}$$

Now from the theory of Fourier cosine series if $f^{(p)}(x)$ is continuous for $|x| \le 1$ and $p = 0, 1, ..., K-1$, and $f^{(K)}(x)$ is integrable

$$|a_r| \le \frac{M}{r^K} \tag{2.143}$$

Since $|T_r(x)| \le 1$ for $|x| \le 1$ the remainder after n terms is $\mathcal{O}\left(\frac{1}{n^{K-1}}\right)$. Now

$$F'(\theta) = -f'(\cos\theta)\sin\theta \tag{2.144}$$

$$F''(\theta) = f''(\cos\theta)\sin^2\theta - f'(\cos\theta)\cos\theta \tag{2.145}$$

$$F'''(\theta) = -f'''(\cos\theta)\sin^3\theta + 3f''(\cos\theta)\sin\theta\cos\theta + f'(\cos\theta)\sin\theta \tag{2.146}$$

and in general, if $F^{(p)}$ is bounded,

$$F^{(p)}(0) = F^{(p)}(\pi) = 0 \tag{2.147}$$

when p is odd, since then $F^{(p)}$ contains the term $\sin\theta$. As a result, the favorable error estimate applies, provided that derivatives of f exist up to the required order.

2.21 Chebyshev Interpolation

We can transform cosine interpolation to interpolation with Chebyshev polynomials by setting

$$G_n(x) = \sum_{r=0}^{n} a_r T_r(x) \tag{2.148}$$

and choosing the coefficients so that

$$G_n(x_j) = f(x_j) \text{ for } j = 0, 1, ...n \tag{2.149}$$

where

$$x_j = \cos\theta_j = \cos\left(\frac{2j+1}{n+1}\frac{\pi}{2}\right) \tag{2.150}$$

These are the zeros of $T_{n+1}(x)$. Now for any $k < n$ we can find the discrete least squares approximation. When $k = n$, because of the above equation, the error is zero, so the least squares approximation is the interpolation polynomial. Moreover, since $G_n(x)$ is a polynomial of degree n, this is the Lagrange interpolation polynomial at the zeros of $T_{n+1}(x)$.

2.22 Portraits of Fourier, Legendre and Chebyshev

Figure 2.1: Joseph Fourier (1768-1830)

Figure 2.2: Adrien-Marie Legendre (1752-1833)

Figure 2.3: Pafnuty Lvovich Chebyshev (1821-1894)