Advanced Computational Fluid Dynamics
AA215A Lecture 4

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Winter Quarter, 2012, Stanford, CA

Abstract

Lecture 4 covers analysis of the equations of gas dynamics

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Analysis of the equations of gas dynamics

Lecture 4
Analysis of the equations of gas dynamics

1.1 Coordinate Transformations

In order to calculate solutions for flows in complex geometric domains, it is often useful to introduce body–fitted coordinates through global, or, as in the case of isoparametric elements, local transformations. With the body now coinciding with a coordinate surface, it is much easier to enforce the boundary conditions accurately. Suppose that the mapping to computational coordinates (ξ_1, ξ_2, ξ_3) is defined by the transformation matrices

$$K_{ij} = \frac{\partial x_i}{\partial \xi_j}, \quad K_{ij}^{-1} = \frac{\partial \xi_i}{\partial x_j}, \quad J = \det(K). \tag{1.1}$$

The Navier-Stokes equations (**??** -**??**) become

$$\frac{\partial}{\partial t}(Jw) + \frac{\partial}{\partial \xi_i} F_i(w) = 0. \tag{1.2}$$

Here the transformed fluxes are

$$F_i = S_{ij} f_j, \tag{1.3}$$

where

$$S = JK^{-1}. \tag{1.4}$$

The elements of S are the cofactors of K, and in a finite volume discretization they are just the face areas of the computational cells projected in the x_1, x_2, and x_3 directions. Using the permutation tensor ϵ_{ijk} we can express the elements of S as

$$S_{ij} = \frac{1}{2} \epsilon_{jpq} \epsilon_{irs} \frac{\partial x_p}{\partial \xi_r} \frac{\partial x_q}{\partial \xi_s}. \tag{1.5}$$

Then

$$\begin{aligned} \frac{\partial}{\partial \xi_i} S_{ij} &= \frac{1}{2} \epsilon_{jpq} \epsilon_{irs} \left(\frac{\partial^2 x_p}{\partial \xi_r \partial \xi_i} \frac{\partial x_q}{\partial \xi_s} + \frac{\partial x_p}{\partial \xi_r} \frac{\partial^2 x_q}{\partial \xi_s \partial \xi_i} \right) \\ &= 0. \end{aligned} \tag{1.6}$$

Defining scaled contravariant velocity components as

$$U_i = S_{ij} u_j, \tag{1.7}$$

the flux formulas may be expanded as

$$F_i = \left\{ \begin{array}{c} \rho U_i \\ \rho U_i u_1 + S_{i1} p \\ \rho U_i u_2 + S_{i2} p \\ \rho U_i u_3 + S_{i3} p \\ \rho U_i H \end{array} \right\}. \tag{1.8}$$

If we choose a coordinate system so that the boundary is at $\xi_l = 0$, the wall boundary condition for inviscid flow is now

$$U_l = 0. \tag{1.9}$$

1.2 Analysis of the equations of gas dynamics: the Jacobian matrices

The Euler equations for the three-dimensional flow of an inviscid gas can be written in integral form, using the summation convention, as

$$\frac{d}{dt}\int_{\Sigma} \boldsymbol{w} dV + \int_{d\Sigma} \boldsymbol{f}_i n_i dS = 0$$

where Σ is the domain, $d\Sigma$ its boundary, $\overrightarrow{n}$ the normal to the boundary, and dV and dS are the volume and area elements. Let x_i, u_i, ρ, p, E and H denote the Cartesian coordinates velocity, density, pressure, energy and enthalpy. In differential form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial}{\partial x_i} \boldsymbol{f}_i(\boldsymbol{w}) = 0 \tag{1.10}$$

where the state and flux vectors are

$$\boldsymbol{w} = \rho \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ E \end{bmatrix}, \quad \boldsymbol{f}_i = \rho u_i \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ 0 \end{bmatrix}$$

Also,

$$p = (\gamma - 1)\rho(E - \frac{u^2}{2}), \quad H = E + \frac{p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}$$

where u is the speed and c is the speed of sound

$$u^2 = u_i^2, c^2 = \frac{\gamma p}{\rho}$$

Let m_i and e denote the momentum components and total energy,

$$m_i = \rho u_i, e = \rho E = \frac{p}{\gamma - 1} + \frac{m_i^2}{2\rho}$$

Then $\boldsymbol{w}$ and f can be expressed as

$$\boldsymbol{w} = \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix}, \quad \boldsymbol{f}_i = u_i \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ u_i \end{bmatrix} \tag{1.11}$$

In a finite volume scheme the flux needs to be calculated across the interface between each pair of cells. Denoting the face normal and area by n_i and S, the flux is FS where

$$\boldsymbol{f} = n_i \boldsymbol{f}_i$$

This can be expressed in terms of the conservative variables $\boldsymbol{w}$ as

$$\boldsymbol{f} = u_n \begin{bmatrix} \rho \\ m_1 \\ m_2 \\ m_3 \\ e \end{bmatrix} + p \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \tag{1.12}$$

where u_n is the normal velocity

$$u_n = n_i u_i = \frac{n_i m_i}{\rho}$$

Also

$$p = (\gamma - 1)(e - \frac{m_i^2}{\rho})$$

In smooth regions of the flow the equations can also be written in quasilinear form as

$$\frac{\partial \boldsymbol{w}}{\partial t} + A_i \frac{\partial \boldsymbol{w}}{\partial x_i} = 0$$

where A_i are the Jacobian matrices

$$A_i = \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}},$$

The composite Jacobian matrix at a face with normal vector $\overrightarrow{n}$ is

$$A = A_i n_i = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{w}}$$

All the entries in $\boldsymbol{f}_i$ and $\boldsymbol{f}$ are homogenous of degree 1 in the conservative variables $\boldsymbol{w}$. It follows that $\boldsymbol{f}_i$ and $\boldsymbol{f}$ satisfy the identities

$$\boldsymbol{f}_i = A_i \boldsymbol{w}, \boldsymbol{f} = A\boldsymbol{w}$$

This is the consequence of the fact that if a quantity q can be expressed in terms of the components of a vector $\boldsymbol{w}$ as

$$q = \prod_j \boldsymbol{w}_j^{\alpha_j}, \text{where} \sum_j \alpha_j = \alpha,$$

then

$$
\begin{aligned}
\sum_i \frac{\partial q}{\partial \boldsymbol{w}_i} \boldsymbol{w}_i &= \sum_i \alpha_i \prod_j \boldsymbol{w}_j^{\alpha_j} \\
&= \sum_i \alpha_i q \\
&= \alpha q
\end{aligned}
\tag{1.13}
$$

In order to evaluate A note that $\frac{\partial u_n}{\partial \boldsymbol{w}}$ and $\frac{\partial p}{\partial \boldsymbol{w}}$ are row vectors

$$\frac{\partial u_n}{\partial \boldsymbol{w}} = \frac{1}{\rho} \left[-u_n, n_1, n_2, n_3, 0\right]$$

and

$$\frac{\partial p}{\partial \boldsymbol{w}} = (\gamma - 1) \left[\frac{u^2}{2}, -u_1, -u_2, -u_3, 1\right]$$

Accordingly

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{w}} = \frac{\partial}{\partial \boldsymbol{w}}(u_n \boldsymbol{w}) + \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[\frac{\partial p}{\partial \boldsymbol{w}}\right] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \end{bmatrix} \left[\frac{\partial u_n}{\partial \boldsymbol{w}}\right] \tag{1.14}$$

Then the Jacobian matrix can be assembled as the sum of a diagonal matrix and two outer products of rank 1,

$$
\begin{aligned}
A &= u_n I + \rho \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} \left[\frac{\partial u_n}{\partial \boldsymbol{w}}\right] + \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[\frac{\partial p}{\partial \boldsymbol{w}}\right] \\
&= u_n I + \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} \left[-u_n, n_1, n_2, n_3, 0\right] + (\gamma - 1) \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[\frac{u^2}{2}, -u_1, -u_2, -u_3, 1\right]
\end{aligned}
$$

Note that every entry in the Jacobian matrix can be expressed in terms of the velocity components u_i and the speed of sound c since

$$H = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}$$

It may also be directly verified that

$$\boldsymbol{f} = A\boldsymbol{w}$$

because u_n is homogeneous of degree 0 with the consequence that $\frac{\partial u_n}{\partial \boldsymbol{w}}$ is orthogonal to $\boldsymbol{w}$.

$$\frac{\partial u_n}{\partial \boldsymbol{w}} \boldsymbol{w} = 0$$

while p is homogenous of degree 1 so that

$$\frac{\partial p}{\partial \boldsymbol{w}} \boldsymbol{w} = p$$

Thus

$$\begin{aligned} A\boldsymbol{w} &= u_n \boldsymbol{w} + \begin{bmatrix} 1 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix} \left[\frac{\partial p}{\partial \boldsymbol{w}}\right] \boldsymbol{w} &= \boldsymbol{f} \end{aligned}$$

Also

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{w}} = A + \frac{\partial A}{\partial \boldsymbol{w}} \boldsymbol{w} = A$$

because $\frac{\partial A}{\partial \boldsymbol{w}}$ is homogeneous of degree 0, with the consequence that $\boldsymbol{w}$ is in the null space of $\frac{\partial A}{\partial \boldsymbol{w}}$.

The special structure of the Jacobian matrix enables the direct identification of its eigenvalues and eigenvectors. Any vector in the 3 dimensional subspace orthogonal to the vectors $\frac{\partial u_n}{\partial \boldsymbol{w}}$ and $\frac{\partial p}{\partial \boldsymbol{w}}$ is an eigenvector corresponding to the eigenvalue u_n, which is thus a triple eigenvalue. It is easy to verify that the vectors

$$r_0 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ \frac{u^2}{2} \end{bmatrix} \quad r_1 = \begin{bmatrix} 0 \\ 0 \\ n_3 \\ -u_2 \\ u_2 n_3 - u_3 n_2 \end{bmatrix} \quad r_2 = \begin{bmatrix} 0 \\ -n_3 \\ 0 \\ n_1 \\ u_3 n_1 - u_1 n_3 \end{bmatrix} \quad r_3 = \begin{bmatrix} 0 \\ n_2 \\ -n_1 \\ 0 \\ u_1 n_2 - u_2 n_1 \end{bmatrix}$$

are orthogonal to both $\frac{\partial u_n}{\partial \boldsymbol{w}}$ and $\frac{\partial p}{\partial \boldsymbol{w}}$. However, r_1, r_2 and r_3 are not independent since

$$\sum_{k=1}^{3} n_k r_k = 0$$

Three independent eigenvectors can be obtained as

$$v_1 = n_1 r_0 + c r_1, v_2 = n_2 r_0 + c r_2, v_3 = n_3 r_0 + c r_3$$

where c is the speed of sound.

In order to verify this note that the middle three elements of $r_k, k = 1, 2, 3$ are equal to $\overrightarrow{i}_k \times \overrightarrow{n}$, where $\overrightarrow{i}_k$ is the unit vector in the k^{th} coordinate direction. Also the last element of v_k is equal to $\overrightarrow{i}_k \cdot (\overrightarrow{n}_k \times \overrightarrow{u})$. If the vectors v_k are not independent they must satisfy the relation of the form

$$\overrightarrow{v} = \sum_{k=1}^{3} \alpha_k \overrightarrow{v}_k = 0$$

for some non zero vector $\overrightarrow{\alpha}$. The first element of $\overrightarrow{v}$ is

$$\sum_{k=1}^{3} \alpha_k n_k = \overrightarrow{\alpha} \cdot \overrightarrow{n}$$

For the next three elements of $\overrightarrow{v}$ to be zero

$$\sum_{k=1}^{3} \alpha_k \left(\overrightarrow{i}_k \times \overrightarrow{n}\right) = \overrightarrow{\alpha} \times \overrightarrow{n} = 0$$

which is only possible if $\overrightarrow{\alpha}$ is parallel to $\overrightarrow{n}$, so that $\overrightarrow{\alpha} \cdot \overrightarrow{n} \neq 0$.

In order to identify the remaining eigenvectors denote the column vectors in A as

$$a_1 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \\ H \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ n_3 \\ u_n \end{bmatrix}$$

Then

$$\rho \frac{\partial u_n}{\partial \boldsymbol{w}} a_1 = 0, \; p \frac{\partial u_n}{\partial \boldsymbol{w}} a_2 = 1$$

$$\frac{\partial p}{\partial \boldsymbol{w}} a_1 = c^2, \; \frac{\partial p}{\partial \boldsymbol{w}} a_2 = 0$$

Now consider a vector of the form

$$r = a_1 + \alpha a_2$$

Then

$$\begin{aligned} Ar &= u_n(a_1 + \alpha a_2) + \alpha a_1 + c^2 a_2 \\ &= \lambda(a_1 + \alpha a_2) = \lambda r \end{aligned} \tag{1.15}$$

if

$$u_n + \alpha = \lambda, \alpha u_n + c^2 = \lambda$$

which is the case if

$$\alpha^2 = c^2$$

Thus the vectors

$$v_4 = a_1 + c a_2, v_5 = a_1 - c a_2$$

are the eigenvectors corresponding to the eigenvalues $u_n + c$ and $u_n - c$. Written in full

$$v_4 = \begin{bmatrix} 1 \\ u_1 + n_1 c \\ u_2 + n_2 c \\ u_3 + n_3 c \\ H + u_n c \end{bmatrix} \quad v_5 = \begin{bmatrix} 1 \\ u_1 - n_1 c \\ u_2 - n_2 c \\ u_3 - n_3 c \\ H - u_n c \end{bmatrix}$$

1.3 Two dimensional flow

The equations of two dimensional flow have the simpler form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \boldsymbol{f}_1(\boldsymbol{w})}{\partial x_1} + \frac{\partial \boldsymbol{f}_2(\boldsymbol{w})}{\partial x_2} = 0$$

where the state and flux vectors are

$$\boldsymbol{w} = \begin{bmatrix} p \\ m_1 \\ m_2 \\ e \end{bmatrix}, \boldsymbol{f}_1 = u_1 \boldsymbol{w} + p \begin{bmatrix} 0 \\ 1 \\ 0 \\ u_1 \end{bmatrix}, \boldsymbol{f}_2 = u_2 \boldsymbol{w} + p \begin{bmatrix} 0 \\ 0 \\ 1 \\ u_2 \end{bmatrix}$$

Now the flux vector normal to an edge with normal components n_1 and n_2 is

$$\boldsymbol{f} = n_1 \boldsymbol{f}_1 + n_2 \boldsymbol{f}_2$$

The corresponding Jacobian matrix is

$$\begin{aligned} A = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{w}} &= n_1 \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{w}} + n_2 \frac{\partial \boldsymbol{f}_2}{\partial \boldsymbol{w}} \\ &= u_n I + \rho \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ H \end{bmatrix} \frac{\partial u_n}{\partial \boldsymbol{w}} + \begin{bmatrix} 0 \\ n_1 \\ n_2 \\ u_n \end{bmatrix} \frac{\partial p}{\partial \boldsymbol{w}} \end{aligned}$$

The eigenvalues of A are

$$u_n, \quad u_n, \quad u_n + c \quad , u_n - c$$

with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ \frac{u^2}{2} \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ -cn_2 \\ cn_1 \\ c(u_2 n_1 - u_1 n_2) \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ u_1 + n_1 c \\ u_2 + n_2 c \\ H + u_n c \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ u_1 - n_1 c \\ u_2 - n_2 c \\ H - u_n c \end{bmatrix}$$

1.4 One dimensional flow

The equations of one dimensional flow are further simplified to

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial}{\partial x} \boldsymbol{f}(\boldsymbol{w}) = 0$$

where the state vector is

$$\boldsymbol{w} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix}$$

and the flux vector is

$$\boldsymbol{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} = u \boldsymbol{w} + \begin{bmatrix} 0 \\ u \\ up \end{bmatrix}$$

Now the Jacobian matrix is

$$A = \frac{\partial f}{\partial \boldsymbol{w}} = uI + \rho \begin{bmatrix} 1 \\ u \\ H \end{bmatrix} \frac{\partial u}{\partial \boldsymbol{w}} + \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} \frac{\partial p}{\partial \boldsymbol{w}} = uI + \begin{bmatrix} 1 \\ u \\ H \end{bmatrix} [-u, 1, 0] + (\gamma - 1) \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix} [\frac{u^2}{2}, -u, 1]$$

The eigenvalues of A are

$$u, u + c, u - c$$

with the corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ u \\ \frac{u^2}{2} \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ u + c \\ H + uc \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ u - c \\ H - uc \end{bmatrix}$$

1.5 Transformation to alternative sets of variables: primitive form

The quasi-linear equations of gas dynamics can be simplified in various ways by transformations to alternative sets of variables.

Consider the equations of three dimensional flow

$$\frac{\partial \boldsymbol{w}}{\partial t} + A_i \frac{\partial \boldsymbol{w}}{\partial x_i} = 0$$

where

$$A_i = \frac{\partial f}{\partial \boldsymbol{w}_i}$$

Under a transformation to new set of variables $\tilde{\boldsymbol{w}}$

$$\tilde{M} \frac{\partial \tilde{\boldsymbol{w}}}{\partial t} + A_i \tilde{M} \frac{\partial \tilde{\boldsymbol{w}}}{\partial x_i} = 0$$

where

$$\tilde{M} = \frac{\partial \boldsymbol{w}}{\partial \tilde{\boldsymbol{w}}}$$

Multiplying by $\frac{\partial \tilde{\boldsymbol{w}}}{\partial \boldsymbol{w}} = M^{-1}$,

$$\frac{\partial \tilde{\boldsymbol{w}}}{\partial t} + \tilde{A}_i \frac{\partial \tilde{\boldsymbol{w}}}{\partial x_i} = 0$$

where

$$\tilde{A}_i = \tilde{M}^{-1} A_i \tilde{M}, A_i = \tilde{M} \tilde{A}_i \tilde{M}^{-1}$$

In the case of the primitive variables,

$$\tilde{\boldsymbol{w}} = \begin{bmatrix} \rho \\ u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix}$$

we find that

$$\tilde{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ u_1 & \rho & 0 & 0 & 0 \\ u_2 & 0 & \rho & 0 & 0 \\ u_3 & 0 & 0 & \rho & 0 \\ \frac{u^2}{2} & u_1 & u_2 & u_3 & \frac{1}{\gamma-1} \end{bmatrix}$$

$$\tilde{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{u_1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{u_2}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{u_3}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u_1 & -(\gamma-1)u_2 & -(\gamma-1)u_2 & \gamma-1 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} u_1 & p & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ 0 & \rho c^2 & 0 & 0 & u_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} u_2 & 0 & p & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & \rho c^2 & 0 & u_2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} u_3 & 0 & 0 & p & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ 0 & 0 & 0 & u_3 & \frac{1}{\rho} \\ 0 & 0 & 0 & \rho c^2 & u_3 \end{bmatrix}$$

1.6 Symmetric form

Consider the equations of one dimensional flow in primitive variables

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \rho c^2\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} = 0$$

Then subtracting the first equation multiplied by c^2 from the third equation, we find that

$$\frac{\partial p}{\partial t} - c^2\frac{\partial \rho}{\partial t} + u(\frac{\partial p}{\partial x} - c^2\frac{\partial \rho}{\partial x}) = 0$$

This is equivalent to a statement that the entropy

$$S = \log(\frac{p}{\rho^\gamma}) = \log p - \gamma \log \rho$$

is constant since

$$dS = \frac{dp}{p} - \gamma\frac{d\rho}{\rho} = \frac{1}{p}(dp - c^2 d\rho)$$

If the entropy is constant, then

$$dp = c^2 d\rho$$

With this substitution the first equation becomes,

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{u}{c^2}\frac{\partial p}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

and now the first two equations can be rescaled as

$$\frac{1}{\rho c}\frac{\partial p}{\partial t} + \frac{u}{\rho c}\frac{\partial p}{\partial x} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{c}{\rho c}\frac{\partial p}{\partial x} + u\frac{\partial u}{\partial x} = 0$$

Thus if we write the equations in terms of the differential variables

$$d\bar{\boldsymbol{w}} = \begin{bmatrix} \frac{dp}{\rho c} \\ du \\ dp - c^2 d\rho \end{bmatrix}$$

we obtain the symmetric form

$$\frac{\partial \bar{\boldsymbol{w}}}{\partial t} + \bar{A}\frac{\partial \bar{\boldsymbol{w}}}{\partial x} = 0$$

where

$$\bar{A} = \begin{bmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{bmatrix}$$

These transformations can be generalized to the equations of three dimensional flow. A convenient scaling is to set

$$d\bar{\boldsymbol{w}} = \begin{bmatrix} \frac{dp}{c^2} \\ \frac{\rho}{c} du_1 \\ \frac{\rho}{c} du_2 \\ \frac{\rho}{c} du_3 \\ \frac{dp}{c^2} - d\rho \end{bmatrix} \tag{1.16}$$

This eliminates the density from the transformation matrices $\bar{M} = \frac{d\boldsymbol{w}}{d\bar{\boldsymbol{w}}}$ and $\bar{M}^{-1} = \frac{d\bar{\boldsymbol{w}}}{d\boldsymbol{w}}$. The equations now take the form

$$\frac{\partial \bar{\boldsymbol{w}}}{\partial t} + \bar{A}_i\frac{\partial \bar{\boldsymbol{w}}}{\partial x_i} = 0 \tag{1.17}$$

where the transformed Jacobian matrices

$$\bar{A}_i = \bar{M}^{-1} A_i \bar{M} \tag{1.18}$$

are simultaneously symmetrized as

$$\bar{A}_1 = \begin{bmatrix} u_1 & c & 0 & 0 & 0 \\ c & u_1 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & u_1 \end{bmatrix} \tag{1.19}$$

$$\bar{A}_2 = \begin{bmatrix} u_2 & 0 & c & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ c & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & u_2 \end{bmatrix} \tag{1.20}$$

$$\bar{A}_3 = \begin{bmatrix} u_3 & 0 & 0 & c & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ c & 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & 0 & u_3 \end{bmatrix} \tag{1.21}$$

while

$$\bar{M} = \frac{\partial \boldsymbol{w}}{\partial \bar{\boldsymbol{w}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ u_1 & c & 0 & 0 & -u_1 \\ u_2 & 0 & c & 0 & -u_2 \\ u_3 & 0 & 0 & c & -u_3 \\ H & cu_1 & cu_2 & cu_3 & -\frac{u^2}{2} \end{bmatrix} \tag{1.22}$$

and

$$\bar{M}^{-1} = \frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{w}} = \begin{bmatrix} \bar{\gamma}\frac{u^2}{2} & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & -\bar{\gamma} \\ -\frac{u_1}{c} & \frac{1}{c} & 0 & 0 & 0 \\ -\frac{u_2}{c} & 0 & \frac{1}{c} & 0 & 0 \\ -\frac{u_3}{c} & 0 & 0 & \frac{1}{c} & 0 \\ \bar{\gamma}(u^2 - H) & -\bar{\gamma}u_1 & -\bar{\gamma}u_2 & -\bar{\gamma}u_3 & \bar{\gamma} \end{bmatrix} \tag{1.23}$$

where

$$\bar{\gamma} = \frac{\gamma - 1}{c^2} \tag{1.24}$$

The combined Jacobian matrix

$$A = n_i A_i \tag{1.25}$$

can now be decomposed as

$$A = n_i \bar{M} \bar{A}_i \bar{M}^{-1} \tag{1.26}$$

Corresponding to the fact that $\bar{A}$ is symmetric one can find a set of orthogonal eigenvectors, which may be normalized to unit length. Then one can express

$$\bar{A} = \bar{V} \Lambda \bar{V}^{-1}$$

where the diagonal matrix Λ contains the eigenvalues $u_n, u, u_n, u_n + c$ and $u_n - c$ as its elements. The matrix $\bar{V}$ containing the corresponding eigenvectors as its columns is

$$\bar{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{n_1}{\sqrt{2}} & \frac{n_1}{\sqrt{2}} & 0 & -n_3 & n_2 \\ \frac{n_2}{\sqrt{2}} & \frac{n_2}{\sqrt{2}} & n_3 & 0 & -n_1 \\ \frac{n_3}{\sqrt{2}} & \frac{n_3}{\sqrt{2}} & -n_2 & n_1 & 0 \\ 0 & 0 & 1 & n_2 & n_3 \end{bmatrix} \tag{1.27}$$

and $\bar{V}^{-1} = \bar{V}^T$. The Jacobian matrix can now be expressed as

$$A = M \Lambda M^{-1} \tag{1.28}$$

where

$$M = \bar{M}\bar{V}, M^{-1} = \bar{V}^T \bar{M}^{-1} \tag{1.29}$$

This decomposition is often useful.
Since

$$c^2 = \frac{dp}{d\rho}$$

it follows that in isentropic flow

$$2c\ dc = \frac{\gamma}{\rho} dp - \frac{\gamma p}{\rho^2} d\rho = \frac{\gamma}{\rho} dp - c^2 d\rho = \frac{\gamma - 1}{\rho} dp$$

Thus

$$\frac{dp}{\rho c} = \frac{2}{\gamma - 1} dc$$

so the equations can also be expressed in the same form, equations (1.17), (1.18), (1.19), (1.20) and (1.21) for the variables

$$\bar{\boldsymbol{w}} = \begin{bmatrix} \frac{2c}{\gamma-1} \\ u_1 \\ u_2 \\ u_3 \\ S \end{bmatrix}$$

with the transformation matrices

$$\bar{M} = \frac{d\boldsymbol{w}}{d\bar{\boldsymbol{w}}} = \begin{bmatrix} \frac{\rho}{c} & 0 & 0 & 0 & -\frac{p}{c^2} \\ \frac{\rho u_1}{c} & \rho & 0 & 0 & -\frac{p u_1}{c^2} \\ \frac{\rho u_2}{c} & 0 & \rho & 0 & -\frac{p u_2}{c^2} \\ \frac{\rho u_3}{c} & 0 & 0 & \rho & -\frac{p u_3}{c^2} \\ \frac{\rho H}{c} & \rho u_1 & \rho u_2 & \rho u_3 & -\rho \frac{p}{c} \frac{u^2}{2} \end{bmatrix}$$

and

$$\bar{M}^{-1} = \frac{d\bar{\boldsymbol{w}}}{d\boldsymbol{w}} = \begin{bmatrix} \bar{\gamma} \frac{u^2}{2} & -\bar{\gamma} u_1 & -\bar{\gamma} u_2 & -\bar{\gamma} u_3 & \bar{\gamma} \\ -\frac{u_1}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\ -\frac{u_2}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\ -\frac{u_3}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\ \frac{\bar{\gamma}}{p}(u^2 - H) & -\frac{\bar{\gamma} u_1}{p} & -\frac{\bar{\gamma} u_2}{p} & -\frac{\bar{\gamma} u_3}{p} & \frac{\bar{\gamma}}{p} \end{bmatrix}$$

where

$$\bar{\gamma} = \frac{\gamma - 1}{\rho c}$$

1.7 Riemann invariants

In the case of one dimensional isentropic flow these equations reduce to

$$\frac{2}{\gamma - 1} \frac{\partial c}{\partial t} + \frac{2u}{\gamma - 1} \frac{\partial c}{\partial x} + c \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{2c}{\gamma - 1} \frac{\partial c}{\partial x} + u \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0$$

Now the first two equations can be added and subtracted to yield

$$\frac{\partial R^+}{\partial t} + (u + c) \frac{\partial R^+}{\partial x} = 0$$

and

$$\frac{\partial R^-}{\partial t} + (u - c) \frac{\partial R^-}{\partial x} = 0$$

where R^+ and R^- are the Riemann invariants

$$R^+ = u + \frac{2c}{\gamma - 1}, R^- = u - \frac{2c}{\gamma - 1}$$

which remain constant as they are transported at the wave speeds $u + c$ and $u - c$. The Riemann invariants prove to be useful in the formulation of far field boundary conditions designed to minimize wave reflection.

1.8 Symmetric hyperbolic form

Equation (1.17) can be written in terms of the conservative variables as

$$\frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial t} + \bar{A}_i \frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial x_i} = 0$$

Here $\frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{w}} = \bar{M}^{-1}$. Now multiplying by $\bar{M}^{T^{-1}}$ the equation is reduced to the symmetric hyperbolic form

$$Q \frac{\partial \boldsymbol{w}}{\partial t} + \hat{A}_i \frac{\partial \boldsymbol{w}}{\partial x_i} = 0 \tag{1.30}$$

where Q is symmetric and positive definite and the matrices $\hat{A}_i$ are symmetric

$$Q = (MM^T)^{-1}, \hat{A}_i = \bar{M}^{T^{-1}} \bar{A}_i \bar{M}^{-1} \tag{1.31}$$

This form could alternatively be derived by multiplying the conservative form (1.2) of the equations by Q. Then

$$\begin{aligned}
Q \frac{\partial \boldsymbol{f}_i}{\partial x_i} &= \bar{M}^{T^{-1}} \bar{M}^{-1} \frac{\partial \boldsymbol{f}}{\partial x_i} \\
&= \bar{M}^{T^{-1}} \bar{M}^{-1} A_i \frac{\partial \boldsymbol{w}}{\partial x_i} \\
&= \bar{M}^{T^{-1}} \bar{M}^{-1} M \bar{A}_i \bar{M}^{-1} \frac{\partial \boldsymbol{w}}{\partial x_i} \\
&= \bar{M}^{T^{-1}} A_i \bar{M}^{-1} \frac{\partial \boldsymbol{w}}{\partial x_i}
\end{aligned}$$

A symmetric hyperbolic form can actually be obtained for any choice of the dependent variables. For example equation (1.17) could be written in terms of the primitive variables as

$$N \frac{\partial \tilde{\boldsymbol{w}}}{\partial t} + \bar{A}_i N \frac{\partial \tilde{\boldsymbol{w}}}{\partial x_i} = 0$$

where

$$N = \frac{\partial \bar{\boldsymbol{w}}}{\partial \tilde{\boldsymbol{w}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{c^2} \\ 0 & \frac{\rho}{c} & 0 & 0 & 0 \\ 0 & 0 & \frac{\rho}{c} & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho}{c} & 0 \\ -1 & 0 & 0 & 0 & \frac{1}{c^2} \end{bmatrix}$$

Then multiplying by N^T we obtain the symmetric hyperbolic form

$$N^T N \frac{\partial \tilde{\boldsymbol{w}}}{\partial t} + N^T \bar{A}_i N \frac{\partial \tilde{\boldsymbol{w}}}{\partial x_i} = 0$$

1.9 Entropy variables

A particular choice of variables which symmetrizes the equations can be derived from functions of the entropy

$$S = \log(\frac{p}{\rho\gamma}) = \log p - \gamma \log \rho \tag{1.32}$$

The last equation of (1.17) is equivalent to the statement that

$$\rho\frac{\partial S}{\partial t} + \rho u_i \frac{\partial S}{\partial x_i} = 0$$

which can be combined with the mass conservation equation multiplied by S

$$S\frac{\partial \rho}{\partial t} + S\frac{\partial}{\partial x_i}(\rho u_i) = 0$$

to yield the entropy conservation law

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u_i S)}{\partial x_i} = 0 \tag{1.33}$$

This is a special case of a generalized entropy function defined as follows. Given a system of conservation laws

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial}{\partial x_i}\boldsymbol{f}_i(\boldsymbol{w}) = 0 \tag{1.34}$$

Suppose that we can find a scalar function $U(\boldsymbol{w})$ such that

$$\frac{\partial U}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}} = \frac{\partial G_i}{\partial \boldsymbol{w}} \tag{1.35}$$

and $U(\boldsymbol{w})$ is a convex function of $\boldsymbol{w}$. Then $U(\boldsymbol{w})$ is an entropy function with an entropy flux $G_i(\boldsymbol{w})$ since multiplying equation (1.34) by $\frac{\partial U}{\partial \boldsymbol{w}}$ we obtain

$$\frac{\partial U}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial U}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial x_i} = 0 \tag{1.36}$$

and using (1.35) this equivalent to

$$\frac{\partial U(\boldsymbol{w})}{\partial t} + \frac{\partial G_i}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial x_i} = 0$$

which is in turn equivalent to the generalized entropy conservation law

$$\frac{\partial U(\boldsymbol{w})}{\partial t} + \frac{\partial}{\partial x_i}G_i = 0 \tag{1.37}$$

Now introduce dependent variables

$$v^T = \frac{\partial U}{\partial \boldsymbol{w}} \tag{1.38}$$

Then the equations are symmetrized. Define the scalar functions

$$Q(v) = v^T \boldsymbol{w} - U(\boldsymbol{w}) \tag{1.39}$$

and

$$R_i(v) = v^T \boldsymbol{f}_i - G_i(\boldsymbol{w}) \tag{1.40}$$

Then

$$\frac{\partial Q}{\partial v} = \boldsymbol{w}^T + v^T \frac{\partial \boldsymbol{w}}{\partial v} - \frac{\partial U}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial v} = \boldsymbol{w}^T$$

and

$$\frac{\partial R_i}{\partial v} = \boldsymbol{f}_i^T + v^T \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial v} - \frac{\partial G_i}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial v} = \boldsymbol{f}_i^T$$

Hence $\frac{\partial \boldsymbol{w}}{\partial v}$ is symmetric.

$$\frac{\partial \boldsymbol{w}_j}{\partial v_k} = \frac{\partial^2 Q}{\partial v_j \partial v_k} = \frac{\partial \boldsymbol{w}_k}{\partial v_j}$$

and $\frac{\partial \boldsymbol{f}_i}{\partial v}$ is symmetric

$$\frac{\partial \boldsymbol{f}_{ij}}{\partial v_k} = \frac{\partial^2 R_i}{\partial v_j \partial v_k} = \frac{\partial \boldsymbol{f}_{ik}}{\partial v_i}$$

and correspondingly

$$\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial v} = A_i \frac{\partial \boldsymbol{w}}{\partial v}$$

is symmetric. Thus the conservation law (1.34) is converted to the symmetric form

$$\frac{\partial \boldsymbol{w}}{\partial v} \frac{\partial v}{\partial t} + A_i \frac{\partial \boldsymbol{w}}{\partial v} \frac{\partial v}{\partial x_i} = 0 \tag{1.41}$$

Multiplying the conservation law (1.34) by $\frac{\partial \boldsymbol{w}}{\partial v}$ from the left produces the alternative symmetric form

$$\frac{\partial v}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial v}{\partial \boldsymbol{w}} A_i \frac{\partial \boldsymbol{w}}{\partial x_i} = 0 \tag{1.42}$$

since

$$\frac{\partial v}{\partial \boldsymbol{w}} A_i = \frac{\partial v}{\partial \boldsymbol{w}} (A_i \frac{\partial \boldsymbol{w}}{\partial v}) \frac{\partial v}{\partial \boldsymbol{w}}$$

Moreover, multiplying (1.34) by v^T recovers equation (1.36) and the corresponding entropy conservation law (1.37).

Conversely, if $U(\boldsymbol{w})$ satisfies the conservation law (1.37), then

$$\frac{\partial U}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial t} = -\frac{\partial G_i}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial x_i}$$

but

$$\frac{\partial U}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial t} = -\frac{\partial U}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial x_i}$$

These can both hold for all $\frac{\partial \boldsymbol{w}}{\partial x_i}$ only if

$$\frac{\partial U}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{w}} = \frac{\partial G_i}{\partial \boldsymbol{w}}$$

so $u(\boldsymbol{w})$ is an entropy function if it is a convex function of $\boldsymbol{w}$.

The property of convexity enables $U(\boldsymbol{w})$ to be treated as a generalized energy, since u becomes unbounded if $\boldsymbol{w}$ becomes unbounded. It can be shown that $-\rho S$ is a convex function of $\boldsymbol{w}$. Accordingly it follows from the entropy equation (1.33) that we can take

$$U(\boldsymbol{w}) = \frac{\rho S}{\gamma - 1} = \frac{\gamma}{\gamma - 1}\rho \log \rho - \frac{\rho}{\gamma - 1} \log p \tag{1.43}$$

as a generalized entropy function where the scaling factor $\frac{1}{\gamma-1}$ is introduced to simplify the resulting expressions.

Using the formula

$$\frac{\partial p}{\partial \boldsymbol{w}} = (\gamma - 1)[\frac{u^2}{2}, -u_1, -u_2, -u_3, 1]$$

we find that

$$v^T = \frac{\partial u}{\partial \boldsymbol{w}} = \left[\frac{\gamma - S}{\gamma - 1}\frac{\rho}{p}\frac{u^2}{2}, \frac{\rho u_1}{p}, \frac{\rho u_2}{p}, \frac{\rho u_3}{p}, \frac{-\rho}{p}\right] \tag{1.44}$$

where

$$u^2 = u_i u_i$$

The reverse transformation is

$$\boldsymbol{w}^T = p\left[-v_5, v_2, v_3, v_4, (1 - \frac{1}{2}(v_2^2 + v_3^2 + v_4^2)v_5\right] \tag{1.45}$$

Now the scalar functions $Q(v)$ and $R_i(v)$ defined by equations (1.39) and (1.40) reduce to

$$Q(v) = \rho, R_i(v) = m_i = \rho u_i \tag{1.46}$$

Accordingly

$$\boldsymbol{w}^T = \frac{\partial p}{\partial v} = \frac{\partial p}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial v}$$

and

$$\boldsymbol{f}_i^T = \frac{\partial m_i}{\partial v} = \frac{\partial m_i}{\partial \boldsymbol{w}}\frac{\partial \boldsymbol{w}}{\partial v}$$

where

$$\frac{\partial \rho}{\partial \boldsymbol{w}} = [1\ 0\ 0\ 0\ 0]$$

and

$$\frac{\partial m_i}{\partial \boldsymbol{w}} = [0\ \delta_{i1}\ \delta_{i2}\ \delta_{i3}\ 0]$$

Thus the first four rows of $\frac{\partial \boldsymbol{w}}{\partial v}$ consist of $\boldsymbol{w}^T, \boldsymbol{f}_1^T, \boldsymbol{f}_2^T$ and $\boldsymbol{f}_3^T$, and correspondingly since $\frac{\partial \boldsymbol{w}}{\partial v}$ is symmetric its first four columns are $\boldsymbol{w}, \boldsymbol{f}_1, \boldsymbol{f}_2$ and $\boldsymbol{f}_3$. The only remaining element is $\frac{\partial \boldsymbol{w}_5}{\partial v_5}$. A direct calculation reveals that

$$\frac{\partial \boldsymbol{w}_5}{\partial v_5} = \rho E^2 + (E + \frac{u^2}{2})p$$

Thus

$$\frac{\partial \boldsymbol{w}}{\partial v} = \begin{bmatrix} \rho & \rho u_1 & \rho u_2 & \rho u_3 & \rho E \\ \rho u_1 & \rho u_1^2 + p & \rho u_2 u_1 & \rho u_3 u_1 & \rho u_1 E + u_1 p \\ \rho u_2 & \rho u_1 u_2 & \rho u_2^2 + p & \rho u_3 u_2 & \rho u_2 E + u_2 p \\ \rho u_3 & \rho u_1 u_3 & \rho u_2 u_3 & \rho u_3^2 + p & \rho u_3 E + u_3 p \\ \rho E & \rho u_1 E + u_1 p & \rho u_2 E + u_2 p & \rho u_3 E + u_3 p & \rho E^2 + (E + \frac{u^2}{2})p \end{bmatrix}$$

$$= \frac{1}{\rho}\boldsymbol{w}\boldsymbol{w}^T + p \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & u_1 \\ 0 & 0 & 1 & 0 & u_2 \\ 0 & 0 & 0 & 1 & u_3 \\ 0 & u_1 & u_2 & u_3 & E + \frac{u^2}{2} \end{bmatrix}$$