Advanced Computational Fluid Dynamics
AA215A Lecture 5

Antony Jameson

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Abstract

Lecture 5 shock capturing schemes for scalar conservation laws

Contents

Lecture 5
Shock Capturing Schemes for Scalar Conservation Laws

1.1 Introduction

A major achievement in the early development of computational fluid dynamics (CFD) was the formulation of non-oscillatory shock capturing schemes. The first such scheme was introduced by Godunov in his pioneering work first published in 1959 (**?**). Godunov also showed that non-oscillatory schemes with a fixed form are limited to first order accuracy. This is not sufficient for adequate engineering simulations. Consequently there were widespread efforts to develop "high resolution" schemes which circumvented Godunov's theorem by blending a second or higher order accurate scheme in smooth regions of the flow with a first order accurate non-oscillatory scheme in the neighborhood of discontinuities. This is typically accomplished by the introduction of logic which detects local extrema and limits their formation or growth. Notable early examples include Boris and Books' flux corrected transport (FCT) scheme published in 1973 (**?**) and Van Leer's Monotone Upstream Conservative Limited (MUSCL) scheme published in 1974 (**?**).

Here we first discuss the formulation of non-oscillatory schemes for scalar conservation laws in one or more space dimension, and illustrate the construction of schemes which yield second order accuracy in the bulk of the flow but are locally limited to first order accuracy at extrema. Next we discuss the formulation of finite volume schemes for systems of equations such as the Euler equations of gas dynamics, and analyze the construction of interface flux formulas with favorable properties such as sharp resolution of discontinuities and assurance of positivity of the pressure and density. The combination of these two ingredients leads to a variety of schemes which have proved successful in practice.

1.2 The need for oscillation control

1.2.1 Odd-even de-coupling

Consider the linear advection equaton for a right running wave

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a > 0. \tag{1.1}$$

Representing the discrete solution at meshpoint by v_j, a semi-discrete scheme with central differences is

$$\frac{dv_j}{dt} + \frac{a}{2\Delta x}\left(v_{j+1} - v_{j-1}\right) = 0 \tag{1.2}$$

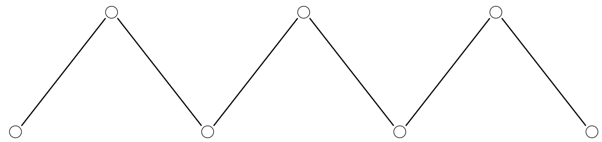

Figure 1.1: Odd-even mode

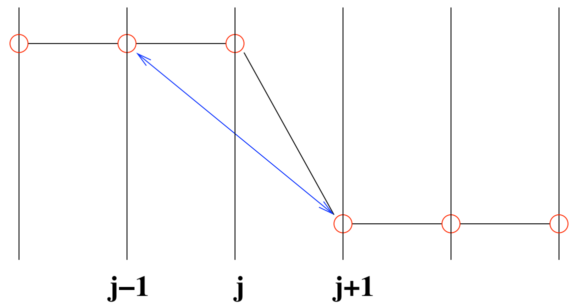

Figure 1.2: Propagation of right running wave

Then an odd-even mode

$$v_j = (-1)^j \tag{1.3}$$

gives

$$\frac{dv_j}{dt} = 0. \tag{1.4}$$

Thus an odd-even mode is a stationary solution, and odd-even decoupling should be removed via the addition of artificial diffusion or upwinding.

1.2.2 Propagation of a step discontinuity

Consider the propagation of a step as a right running wave by the central difference scheme. Now the discrete derivative

$$D_x v_j = \frac{v_{j+1} - v_{j-1}}{2\Delta x} < 0 \tag{1.5}$$

and hence with $a > 0$

$$\frac{dv_j}{dt} > 0 \tag{1.6}$$

giving an overshoot. On the other hand the upwind scheme

$$D_x^- v_j = \frac{v_j - v_{j-1}}{\Delta x} \tag{1.7}$$

correctly yields

$$\frac{dv_j}{dt} = 0. \tag{1.8}$$

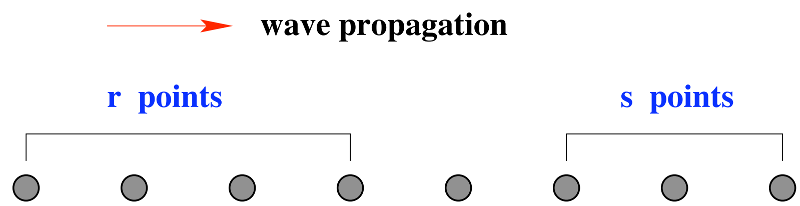

Figure 1.3: Barrier theorem

1.3 Iserles' barrier theorem

Both the need to suppress odd-even modes and the need to prevent overshoots in the propagation of a step discontinuity motivate the use of an upwind scheme. However purely upwind schemes are also subject to limitations as a consequence of Iserles' barrier theorem (**?**).

Consider the approximation of the linear advection equation by a semi-discrete scheme with r upwind points and s downwind points. The theorem states that the maximum order of accuracy of a stable scheme is

$$\min(r+s, 2r, 2s+2) \tag{1.9}$$

This is a generalization of an earlier result of Engquist and Osher that the maximum order of a accuracy of a stable upwind semi-discrete scheme is two ($s = 0$ in formula (1.9)). It may also be compared to Dahquist's result that A-stable linear multistep schemes for ODEs are at most second order accurate. One may conclude from Iserles' theorem that upwind biased schemes may be preferred over purely upwind schemes as a route to attaining higher order accuracy.

1.4 Stability in the L_∞ norm

Consider the nonlinear conservation law for one dependent variable with diffusion

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial}{\partial x}\left(\mu(u)\frac{\partial u}{\partial x}\right), \quad \mu \geq 0. \tag{1.10}$$

With zero diffusion (1.10) is equivalent in smooth regions to

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0 \tag{1.11}$$

where the wave speed is

$$a(u) = \frac{\partial f}{\partial u}.$$

The solution is constant along characteristics

$$x - a(u)t = \xi$$

so extrema remain unchanged as they propagate unless the characteristics converge to form a shock wave, a process that does not increase extrema. With a positive diffusion coefficient the right hand side of equation (1.10) is negative at a maximum and positive at a minimum. Thus in a true solution of (1.10) extrema do not increase in absolute value. It follows that L_∞ stability is an appropriate criterion for discrete schemes consistent with the properties of true solutions of the nonlinear conservation law (1.10).

Consider the general discrete scheme

$$v_i^{n+1} = \sum_j a_{ij} v_j^n \tag{1.12}$$

where the solution at time level $n+1$ depends on the solution over an arbitrarily large stencil of points at time level n. A Taylor series expansion of (1.12) about the point x_i at time t yields

$$\begin{aligned} v(x_i) + \Delta t \frac{\partial v_j}{\partial t}(x_i) + \frac{\Delta t^2}{2}\frac{\partial^2 v_j}{\partial t^2}(x_i) = \\ \sum_j a_{ij}\left[v(x_i) + (x_j - x_i)\frac{\partial v(x_i)}{\partial x} + \frac{(x_j - x_i)^2}{2}\frac{\partial^2 v(x_i)}{\partial x^2} + ...\right] \end{aligned} \tag{1.13}$$

For consistency with any equation with no source term, as is the case for equation (1.10),

$$\sum_j a_{ij} = 1 \tag{1.14}$$

Also it follows from (1.12) that

$$|v_i^{n+1}| \leq \sum_j |a_{ij}||v_j^n| \leq \sum_j |a_{ij}| \; \|v^n\|_\infty \tag{1.15}$$

and hence

$$\|v^{n+1}\|_\infty \leq \max_i \sum_j |a_{ij}| \|v^n\|_\infty \tag{1.16}$$

where equality is realized if

$$v_j^n = \text{sign}(a_{ij})$$

for the row for which $\sum_j |a_{ij}|$ has its maximum value. Thus for L_∞ stability

$$\max_i \sum_j |a_{ij}| \leq 1 \tag{1.17}$$

If one writes (1.12) in matrix vector form

$$v^{n+1} = Av^n,$$

this may be recognized as the condition that the induced matrix norm

$$\|A\|_\infty = \sup_v \frac{\|Av\|_\infty}{\|v\|_\infty} \leq 1 \tag{1.18}$$

Now (1.14) and (1.17) together imply

$$a_{ij} \geq 0 \tag{1.19}$$

because if any $a_{ij} < 0$, then on taking absolute values in (1.14) the sum would have a value > 1. Therefore the discrete scheme (1.12) is L_∞ stable if and only if the coefficients a_{ij} are non-negative. Note that this also implies that given initial data that is everywhere non-negative, the discrete scheme (1.12) has the "positivity" property that it will preserve a non-negative solution at every time step.

Using the consistency condition (1.14), the discrete scheme can be written as

$$\begin{aligned} v_i^{n+1} &= \left(\sum_j (a_{ij}) \right) v_i^n + \sum_{j \neq i} a_{ij}(v_j^n - v_i^n) \\ &= v_i^n + \Delta t \sum_{j \neq i} \tilde{a}_{ij}(v_j^n - v_i^n) \end{aligned} \tag{1.20}$$

where

$$a_{ij} = \Delta t(\tilde{a}_{ij}).$$

This displays the scheme as a forward Euler time stepping scheme. Moreover, comparing (1.20) with (1.12)

$$a_{ii} = 1 - \Delta t \sum_{j \neq i} \tilde{a}_{ij} \tag{1.21}$$

Thus (1.19) can only be satisfied if

$$\tilde{a}_{ij} \geq 0$$

and the time step also satisfies the constraint

$$\Delta t \leq \frac{1}{\sum_{j \neq i} \tilde{a}_{ij}}$$

This is a generalization of the Courant-Friedrichs-Lewy (CFL) condition for schemes with an arbitrary stencil. In the case of linear advection

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0,$$

the simplest L_∞ stable scheme is the upwind scheme

$$\begin{aligned} v_j^{n+1} &= v_j^n - \lambda(v_j^n - v_{j-1}^n) \\ &= (1 - \lambda) v_j^n + \lambda v_{j-1}^n \end{aligned} \tag{1.22}$$

where λ is the CFL number $\frac{a \Delta t}{\Delta x}$ and $0 \leq \lambda \leq 1$. Here v^{n+1} is a convex combination of v_i^n and v_{i-1}^n. When $\lambda = 1$ this scheme exactly represents propagation along the characteristics.

It should also be noted that the discussion in this section is not limited to the one dimensional problem, and applies equally to the case of multi-dimensional equations discretized on arbitrary unstructured grids.

1.5 Local extremum diminishing (LED) schemes

L_∞ stability does not exclude the possibility that a monotonically decreasing profile (figure 1.4(a)) could develop into an oscillatory profile (figure 1.4(b)). This motivates the stricter requirement that local extrema cannot grow in the numerical solution, nor can new local extrema be created. Such a scheme will be called local extremum diminishing (LED).

Figure 1.4: Local oscillation

Consider now the discrete scheme

$$v_i^{n+1} = \sum_j a_{ij} v_j^n \tag{1.23}$$

where the coefficients $a_{ij} \neq 0$ only for the nearest neighbors to the mesh point i, in one or more space dimensions. The argument of the previous section can now be repeated where the summations are now limited to the nearest neighbors of the mesh point i, denoted by the set N_i. For consistency with an equation with no source term

$$\sum_{j \in N_i} a_{ij} = 1, \tag{1.24}$$

while for no increase in $|v_i|$

$$\sum_{j \in N_i} |a_{ij}| \leq 1. \tag{1.25}$$

Thus the scheme is LED if for all i

$$\begin{aligned} a_{ij} &\geq 0, \\ a_{ij} &= 0 \quad \text{if } j \text{ is not a neighbor of } i \end{aligned} \tag{1.26}$$

1.6 Total variation diminishing (TVD) schemes

A useful measure of the oscillation of a one dimensional function $u(x)$ is its total variation

$$TV(u) = \int_a^b \left| \frac{\partial u}{\partial x} \right| dx. \tag{1.27}$$

Correspondingly the total variation of a discrete solution, say v_j, $j = 0, 1, ..., n$, is defined as

$$TV(v) = \sum_{j=1}^{n} |v_{j+1} - v_j|. \tag{1.28}$$

It was proposed by Harten (**?**) that a good recipe for the construction of non-oscillatory shock capturing schemes is to require that the total variation of the discrete solution cannot increase. Such schemes are called total variation diminishing (TVD). This criterion has been widely used and also extended to the concept of total variation bounded (TVB) schemes which permit only a bounded increase in the total variation.

Let x_j, $j = 1, 2, ..., n-1$, be the interior extrema of $u(x)$, and augment there with the endpoints $x_0 = a$ and $x_n = b$. Referring to figure 1.5, it can be seen that over an interval in which $u(x)$ is

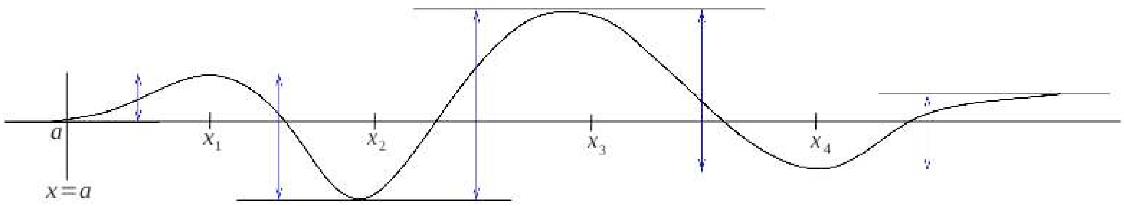

Figure 1.5: Equivalent LED and TVD schemes (one dimensional case)

increasing, say x_{j-1} to x_j, the contribution to $TV(u)$ is $u(x_j) - u(x_{j-1})$, while over the following interval in which $u(x)$ is decreasing the contribution to $TV(u)$ is $u(x_j) - u(x_{j+1})$. Thus

$$\begin{aligned} TV(u) &= 2\sum(\text{interior maxima}) \\ &\quad -2\sum(\text{interior minima}) \\ &\quad \pm(\text{the end values}). \end{aligned} \tag{1.29}$$

Similarly in the discrete case

$$\begin{aligned} TV(v) &= 2\sum(\text{interior maxima}) \\ &\quad -2\sum(\text{interior minima}) \\ &\quad \pm v_0 \pm v_n. \end{aligned} \tag{1.30}$$

If the initial data is, say, monotonically decreasing the total variation is

$$TV(v^0) = v_0^0 - v_n^0.$$

If the end values are fixed the introduction of any interior extrema will produce an increase in the total variation. Thus a TVD scheme preserves the monotonicity of initial data which is monotonic. Also if the end values are fixed the discrete scheme will be TVD if interior maxima cannot increase, interior minima cannot decrease and no new extrema are introduced. Accordingly LED schemes are TVD. The converse is not necessarily true. This is illustrated in figure 1.6, in which the upper and lower profiles have the same total variation with a value equal to 4. Thus a TVD scheme would allow a shift from the lower to the upper profile, which would not be permitted by an LED scheme because it would incur the formation of a new maximum at C'. In this sense the LED criterion is more stringent than the TVD criterion, while still consistent with the properties of the nonlinear conservation law (1.10).

Harten derived conditions for the coefficients of a 3 point TVD scheme in one dimension that are essentially equivalent to the positivity condition (1.26) for an LED scheme. The conditions for a multipoint TVD scheme derived by Jameson and Lax (**?**) would be hard to realize in practice. The LED criterion is directly applicable to multi-dimensional discretizations on both structured and unstructured meshes whereas the use of $\int_D \|\nabla u\|\, dS$ as a measure of total variation can lead to anomalous results on a triangular mesh using any of the standard norms (**?**). For the foregoing reasons the analysis in the following sections is based on the LED principle.

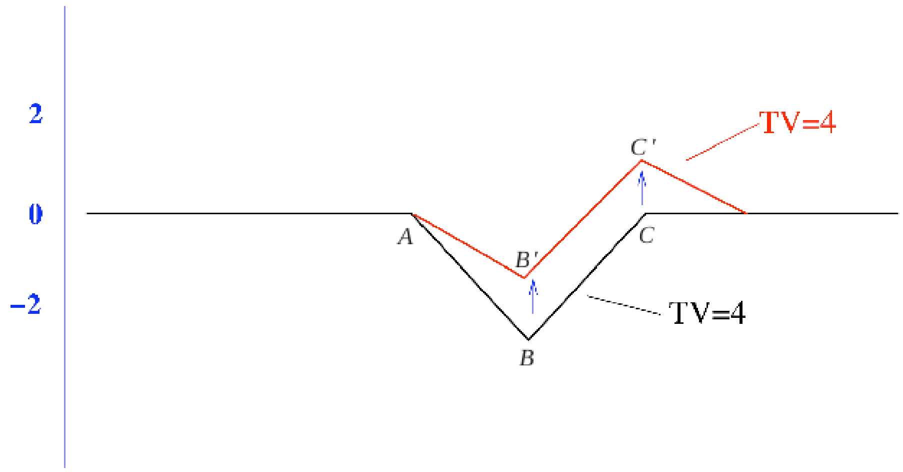

Figure 1.6: Distinction between LED and TVD schemes

1.7 Semi-discrete L_∞ stable and LED schemes

It is often convenient to separate the formulation of the time stepping and space discretization scheme by first using a semi-discretization to reduce the continuous equation to a set of ordinary differential equations with the general form

$$\frac{dv_i}{dt} = \sum_j a_{ij} v_j. \tag{1.31}$$

or in matrix vector notation

$$\frac{dv}{dt} = Av. \tag{1.32}$$

Assuming that the discrete values $v_i(t)$ correspond to a sufficiently smooth function $v(x,t)$ at $x = j\Delta x$, a Taylor series expansion yields

$$\frac{dv_i}{dt} = \sum_j a_{ij} \left[v(x_i) + (x_j - x_i)\frac{\partial v(x_i)}{\partial x} + \left(\frac{x_j - x_i}{2}\right)^2 \frac{\partial^2 v(x_i)}{\partial x^2} + ... \right] \tag{1.33}$$

Accordingly the semi-discrete equation (1.31) is consistent with a differential equation with no source term only if

$$\sum_j a_{ij} = 0. \tag{1.34}$$

Then equation (1.31) can be written as

$$\frac{dv_i}{dt} = \left(\sum_j a_{ij}\right) v_i + \sum_{j \neq i} a_{ij}(v_j - v_i) \tag{1.35}$$

or

$$\frac{dv_i}{dt} = \sum_{j \neq i} a_{ij}(v_j - v_i) \tag{1.36}$$

Suppose that

$$a_{ij} \geq 0, \quad j \neq i \tag{1.37}$$

Then if v_i is a maximum

$$v_j - v_i \leq 0 \tag{1.38}$$

and

$$\frac{dv_i}{dt} \leq 0 \tag{1.39}$$

Similarly if v_i is a minimum

$$v_j - v_i \geq 0 \tag{1.40}$$

and

$$\frac{dv_i}{dt} \geq 0 \tag{1.41}$$

Now $\|v\|_\infty$ can increase only if the maximum increases or the minimum decreases. But

$$\left|\frac{dv_i}{dt}\right| \leq \|A\|_\infty \|v\|_\infty \tag{1.42}$$

so if $|v_i| < \|v\|_\infty$ there is a time interval $\in > 0$ during which it cannot become an extremum, while if $|v_i| = \|v\|_\infty$ it follows from (1.39) or (1.41) that

$$\frac{d}{dt}|v_i| \leq 0 \tag{1.43}$$

Thus condition (1.37) is sufficient to ensure that $\|v\|_\infty$ does not increase.

Suppose that condition (1.37) is not satisfied. Then choosing $v_i = 1$, $v_j = 1$ if $a_{ij} \geq 0$, $v_j = 0$ if $a_{ij} < 0$, one obtains

$$\frac{dv_i}{dt} > 0 \tag{1.44}$$

and $\|v\|_\infty$ will increase. Accordingly the semi-discrete scheme (1.31) is L_∞ stable if and only if the positivity condition (1.37) is satisfied.

As in the case of the discrete scheme, the semi-discrete scheme will be LED if it is limited to a compact stencil of nearest neighbors with non-negative coefficients $a_{ij} \geq 0$, $a_{ij} = 0$ if j is not a neighbor of i because then the same argument can be repeated while examining the behavior of v_i with respect only to its neighbors.

1.8 Growth of the L_∞ norm with a source term

Consider the general semi-discrete scheme

$$\frac{dv_i}{dt} = \sum_j a_{ij} v_j \tag{1.45}$$

where

$$\sum_j a_{ij} = \alpha.$$

It now follows from a Taylor series expansion as in section 1.7, that the semi-discrete is constant with the differential equation

$$\frac{du}{dt} = Lu + \alpha u \tag{1.46}$$

where L is a source free differential operator. Subtracting $\left(\sum_j a_{ij} - \alpha\right) v_i$ the scheme can be written with no loss of generality as

$$\frac{dv_i}{dt} = \alpha v_i + \sum_{j \neq i} a_{ij}(v_j - v_i). \tag{1.47}$$

Set

$$v_i = w_i e^{\alpha t}. \tag{1.48}$$

Then

$$\frac{dv_i}{dt} = \left(\frac{dw_i}{dt} + \alpha w_i\right) e^{\alpha t}. \tag{1.49}$$

Consequently

$$\frac{dw_i}{dt} = \sum_{j \neq i} a_{ij}(w_j - w_i). \tag{1.50}$$

Therefore, if $a_{ij} \geq 0$, $\|w\|_\infty$ does not increase, and the growth of $\|v\|_\infty$ is bounded by $e^{\alpha t}$.

1.9 Accuracy limitation on L_∞ stable and LED schemes

Suppose that the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1.51}$$

is approximated on a uniform grid by a fixed scheme with the same coefficients at every mesh point

$$v_j^{n+1} = \sum_{k=-\infty}^{\infty} c_k v_{j+k}^n. \tag{1.52}$$

A Taylor series expansion now yields

$$v(x_j) + \Delta t \frac{\partial v}{\partial t}(x_j) + \frac{\Delta t^2}{2}\frac{\partial^2 v}{\partial t^2}(x_j) = \sum_{k=-\infty}^{\infty} c_k \left[v(x_i) + k\Delta x \frac{\partial v(x_i)}{\partial x} + \frac{k^2 \Delta x^2}{2}\frac{\partial^2 v(x_i)}{\partial x^2} + ...\right] \tag{1.53}$$

In order to realize second order accuracy the coefficients c_k must satisfy the conditions

$$\sum_k c_k = 1 \tag{1.54}$$

$$\sum_k k c_k = -\lambda \tag{1.55}$$

$$\sum_k k^2 c_k = \lambda^2 \tag{1.56}$$

where

$$\lambda = a\frac{\Delta t}{\Delta x}.$$

Since $c_k \geq 0$ we can set

$$\alpha_k = \sqrt{c_k}, \quad \beta_k = k\sqrt{c_k} = k\alpha_k \tag{1.57}$$

Then

$$\sum_k \alpha_k^2 = 1 \tag{1.58}$$

$$\sum_k \alpha_k \beta_k = -\lambda \tag{1.59}$$

$$\sum_k \beta_k^2 = \lambda^2 \tag{1.60}$$

and consequently

$$\left(\sum_k \alpha_k \beta_k\right)^2 = \left(\sum_k \alpha_k^2\right)\left(\sum_k \beta_k^2\right). \tag{1.61}$$

But by the Cauchy-Schwarz inequality

$$\left(\sum_k \alpha_k \beta_k\right)^2 \leq \left(\sum_k \alpha_k^2\right)\left(\sum_k \beta_k^2\right). \tag{1.62}$$

with equality only if the vectors α and β are aligned, or for some scale r, $\beta_k = r\alpha_k$ for all k. According to the definition (1.57) of α_k and β_k, it follows that there can only be one nonzero c_k. Moreover it follows from (1.54) that this must have the value unity. Taking

$$\begin{aligned} c_k &= 0, \qquad k \neq -1 \\ c_{-1} &= 1 \end{aligned} \tag{1.63}$$

we recover the standard upwind scheme

$$v_j^{n+1} = v_j^n - \lambda(v_j^n - v_{j-1}^n) \tag{1.64}$$

with a CFL number $\lambda = 1$, for which the scheme corresponds to exact propagation along characteristics. A choice such as $c_{k-2} = 1$ corresponds to the upwind scheme applied over double intervals with $\lambda = 1$. It is evident that no solution is possible for time steps such that $\lambda \neq 1$.

We conclude that it is not possible for a discrete scheme with fixed non-negative coefficients at every mesh point to yield better than first order accuracy. In order to overcome this barrier it is necessary to consider schemes in which the discretization is locally adapted to the solution. Typically these schemes enforce positivity or the LED principle only in the neighborhood of extrema which can be detected by a change of sign in the slope measured by $\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$. The earliest examples of schemes of this type are Boris and Book's flux corrected transport (FCT) scheme (**?**) and Van Leer's monotone upstream conservative limited (MUSCL) scheme (**?**).

1.10 Artificial diffusion and upwinding

Taking as the simplest possible example the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1.65}$$

which represents right running waves of the form

$$u(x,t) = f(x - at) \tag{1.66}$$

when $a > 0$, an upwind semi-discretization is

$$\frac{dv_j}{dt} + \frac{a}{\Delta x}(v_j - v_{j-1}) = 0. \tag{1.67}$$

This can be equivalently written as

$$\frac{dv_j}{dt} + \frac{1}{2}\frac{a}{\Delta x}(v_{j+1} - v_{j-1}) - \frac{1}{2}\frac{a}{\Delta x}(v_{j+1} - 2v_j + v_{j-1}) = 0 \tag{1.68}$$

in which $\frac{\partial u}{\partial x}$ is approximated by a second order accurate central difference formula modified by an approximation to $-\frac{1}{2}a\Delta x\frac{\partial^2 u}{\partial x^2}$. Thus upwinding is equivalent to the addition of artificial diffusion with a coefficient proportional to $a\frac{\Delta t}{\Delta x}$, the mesh width multiplied by half the wave speed.

We may consider a class of schemes with artificial diffusion of the form

$$\frac{dv_j}{dt} + \frac{a}{2\Delta x}(v_{j+1} - v_{j-1}) - \frac{\alpha a}{2\Delta x}(v_{j+1} - 2v_j + v_{j-1}) = 0 \tag{1.69}$$

where α is a parameter to be chosen. This can be written in conservative form as

$$\Delta x\frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0 \tag{1.70}$$

where the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}a(v_{j+1} + v_j) - \frac{1}{2}\alpha a(v_{j+1} - v_j). \tag{1.71}$$

Then

$$h_{j+\frac{1}{2}} = av_j - \frac{1}{2}(\alpha - 1)a(v_{j+1} - v_j) \tag{1.72}$$

and

$$h_{j-\frac{1}{2}} = av_j - \frac{1}{2}(\alpha + 1)a(v_j - v_{j-1}) \tag{1.73}$$

so

$$\Delta x\frac{dv_j}{dt} = \frac{1}{2}(\alpha - 1)(v_{j+1} - v_j) + \frac{1}{2}(\alpha + 1)a(v_{j-1} - v_j) \tag{1.74}$$

and the scheme is LED if $\alpha \geq 1$.

We can also consider the energy stability of this scheme. Suppose (1.65) holds in the interval $0 \leq x \leq L$, and it is solved with $u(0,t)$ specified and $u(L,t)$ free, which are the proper boundary conditions for a right running wave. Multiplying by u and integrating from 0 to L,

$$\begin{aligned} \int_0^L u\frac{\partial u}{\partial t}dx &= \frac{d}{dt}\int_0^L \frac{u^2}{2}dx \\ &= -a\int_0^L u\frac{\partial u}{\partial x}dx \\ &= \frac{1}{2}a\left(u(0)^2 - u(L)^2\right). \end{aligned} \tag{1.75}$$

If $u(0,t)$ is fixed equal to zero, the energy decays if $u(L) \neq 0$.

Suppose the semi-discretization is over n equal intervals from $x_0 = 0$ to $x_n = L$. Let v_0 be fixed corresponding to the inflow boundary condition. The outflow boundary value v_n should depend on the solution. To complete the semi-discrete scheme at x_n, we use linear extrapolation to establish a value at an extra mesh point x_{n+1}, $v_{n+1} = v_n$. Now multiplying (1.70) by v_j and summing by parts we find that

$$\begin{aligned}\frac{dE}{dt} &= \Delta x \sum_{j=0}^{n} v_j \frac{dv_j}{dt} \\ &= -\sum_{j=1}^{n} v_j (h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \\ &= h_{\frac{1}{2}} v_1 - \sum_{j=1}^{n-1} h_{j+\frac{1}{2}} (v_{j+1} - v_j) - h_{n+\frac{1}{2}} v_n. \end{aligned} \tag{1.76}$$

In the case $\alpha = 0$, now

$$\frac{dE}{dt} = \frac{1}{2} a v_0 v_1 - \frac{1}{2} a v_n^2 \tag{1.77}$$

and if $v_0 = 0$ the discrete energy decays if $v_n \neq 0$ as in the case of the true solution.

If $\alpha > 0$, $v_0 = 0$ and $v_{n+1} = v_n$

$$\frac{dE}{dt} = -\frac{1}{2} a v_n^2 - \frac{1}{2} \alpha a v_1^2 - \frac{1}{2} \alpha a \sum_{j=1}^{n-1} (v_{j+1} - v_j)^2 \tag{1.78}$$

Accordingly there is a negative contribution to $\frac{dE}{dt}$ from every interior interface. Thus finally the semi-discrete scheme is neutrally energy stable if $\alpha = 0$, energy stable if $\alpha > 0$, and LED if $\alpha \geq 1$.

1.11 Artificial Diffusion and LED Schemes for Nonlinear Conservation Laws

We now consider scalar nonlinear conservation laws of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.79}$$

in the interval $0 \leq x \leq L$, where the boundary values $u(0)$ and $u(L)$ are specified at an inflow boundary and free at an outflow boundary. We can write the conservation law in integral forms as

$$\frac{d}{dt} \int_0^L u dx + f(u(L)) - f(u(0)) = 0 \tag{1.80}$$

Suppose that the domain is divided into a uniform grid of cells of width Δx_j where cell j extends from $x_j - \frac{1}{2}\Delta x$ to $x_j + \frac{1}{2}\Delta x$. Applying (1.80) separately to each cell we obtain the semi-discrete finite volume scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$

where v_j represents the average value of u in cell j, and $h_{j+\frac{1}{2}}$ is the numerical flux across the interface separating cells j and $j+1$. Introducing artificial diffusion, we define the numerical fluxes

$$h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - \alpha_{j+\frac{1}{2}}(v_{j+1} - v_j)$$

where

$$f_j = f(v_j)$$

and $\alpha_{j+\frac{1}{2}}$ is the coefficient of artificial diffusion. Define also a numerical estimate of the wave speed $a(u) = \frac{\partial f}{\partial u}$ as

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1}-f_j}{v_{j+1}-v_j}, & v_{j+1} \neq v_j \\ \frac{\partial f}{\partial u}|_{v_j}, & v_{j+1} = v_j \end{cases}$$

Now the numerical fluxes can be expressed as

$$h_{j+\frac{1}{2}} = f_j + \frac{1}{2}(f_{j+1} - f_j) - \alpha_{j+\frac{1}{2}}(v_{j+1} - v_j) = f_j - (\alpha_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}})(v_{j+1} - v_j)$$

and

$$h_{j-\frac{1}{2}} = f_j - \frac{1}{2}(f_j - f_{j-1}) - \alpha_{j-\frac{1}{2}}(v_j - v_{j-1}) = f_j - (\alpha_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}})(v_j - v_{j-1})$$

Thus the semi-discrete scheme reduces to

$$\Delta x \frac{dv_j}{dt} = (\alpha_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}})(v_{j+1} - v_j) + (\alpha_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}})(v_{j-1} - v_j)$$

Accordingly it is LED if for all j

$$\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|$$

.

1.12 The First Order Upwind Scheme

The least diffusive LED scheme is obtained by setting

$$\alpha_{j+\frac{1}{2}} = \frac{1}{2}|a_{j+\frac{1}{2}}|$$

to produce the diffusive flux

$$d_{j+\frac{1}{2}} = \frac{1}{2}|a_{j+\frac{1}{2}}|\Delta v_{j+\frac{1}{2}}$$

where

$$v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

This is the pure first order upwind scheme since if $a_{j+\frac{1}{2}} > 0$

$$d_{j+\frac{1}{2}} = \frac{1}{2}\frac{f_{j+1} - f_j}{v_{j+1} - v_j}(v_{j+1} - v_j) = \frac{1}{2}(f_{j+1} - f_j)$$

and

$$h_{j+\frac{1}{2}} = f_j$$

while if $a_{j+\frac{1}{2}} < 0$

$$h_{j+\frac{1}{2}} = f_{j+1}$$

Thus the upwind scheme is the least diffusive first order accurate LED scheme.

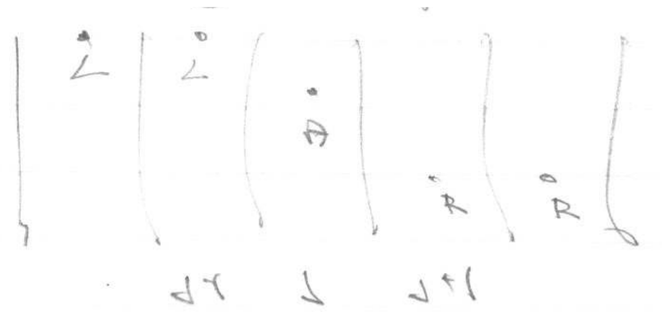

Figure 1.7: Shock structure

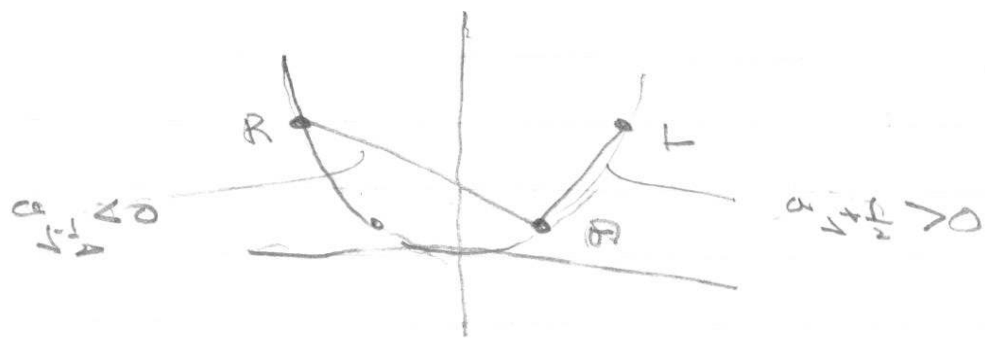

Figure 1.8: Convex flux

1.13 Shock Structure of the Upwind Scheme

The first order upwind scheme supports a numerical structure of a stationary shock with a single interior point, as illustrated in Figure 1.7, where the upstream and downstream values are denoted by the subscripts L and R, and the transitional value located at cell j is denoted by the subscript A. The jump condition for a shock moving at a speed S is

$$f_R - f_L = S(u_R - u_L)$$

For a stationary shock $f_R = f_L$, and since the characteristics converges on a shock the true wave speed changes sign. This is also the case for the numerical wave speed. Assuming $f(u)$ is a convex function of u, as illustrated in Figure 1.8

$$a_{j-\frac{1}{2}} = \frac{f_A - f_L}{v_A - v_L} > 0, a_{j+\frac{1}{2}} = \frac{f_R - f_A}{v_R - v_A} < 0,$$

Specifically in the case of Burgers' equation

$$f(u) = \frac{u^2}{2}, v_R = -v_L, a_{j-\frac{1}{2}} = \frac{1}{2}(v_A + v_L) > 0, a_{j+\frac{1}{2}} = \frac{1}{2}(v_R + v_A) < 0$$

Accordingly the numerical fluxes are

$$h_{j-\frac{1}{2}} = f_{j-1} = f_L, h_{j+\frac{1}{2}} = f_{j+1} = f_R$$

while the interfaces between cells to the left of j are equal to f_L, and those between cells to the right of j are equal to f_R. Hence

$$\frac{d}{dt}v_{j-1} = \frac{f_L - f_L}{\Delta x} = 0$$
$$\frac{d}{dt}v_j = \frac{f_R - f_L}{\Delta x} = 0$$
$$\frac{d}{dt}v_{j+1} = \frac{f_R - f_R}{\Delta x} = 0$$

With an appropriate choice of the numerical flux a single point numerical structure can also be obtained for stationary shock in gas dynamics. Notice also the case where the wave speed changes sign from negative to positive

$$\alpha_{j-\frac{1}{2}} < 0, \alpha_{j+\frac{1}{2}} > 0$$

This represents an expansion region, comparable to the inflow boundary of the supersonic zone in a transonic flow. Now

$$h_{j-\frac{1}{2}} = f_j, h_{j+\frac{1}{2}} = f_j$$

and

$$\frac{dv_j}{dt} = 0$$

1.14 Upwinding and Conservation

In early formulation of upwind schemes the scheme itself was switched according to the sign of the wave speed

$$\frac{dv_j}{dt} + \frac{f_j - f_{j-1}}{\Delta x} = 0, a_j > 0$$
$$\frac{dv_j}{dt} + \frac{f_{j+1} - f_j}{\Delta x} = 0, a_j < 0$$

Suppose that there is a transition between j and j+1

$$a_k > 0, k \leq j$$

$$a_k < 0, k \geq j+1$$

Now

$$\frac{dv_j}{dt} + \frac{f_j - f_{j-1}}{\Delta x} = 0$$

corresponding to a numerical flux

$$h_{j+\frac{1}{2}} = f_j$$

while

$$\frac{dv_{j+1}}{dt} + \frac{f_{j+2} - f_{j+1}}{\Delta x} = 0$$

corresponding to

$$h_{j+\frac{1}{2}} = f_{j+1}$$

Accordingly the scheme is not conservative, since the interior fluxes do not cancel when the solution values are summed over j. In fact, with f_0 and f_n fixed since both boundaries are inflow boundaries

$$\Delta x \sum_{j=1}^{n-1} \frac{dv_j}{dt} = -f_1 + f_0 - f_2 + f_1 ... - f_j + f_{j-1} - f_{j+2} + f_{j+1} ... - f_n + f_{n-1} = f_0 - f_n - f_j + f_{j+1}$$

whereas the true solution satisfies

$$\frac{d}{dt}\int_{x_0}^{x_n} u dx = -\int_{x_0}^{x_n} \frac{\partial f}{\partial x} dx = f(u(x_0)) - f(u(x_n))$$

By upwinding the flux rather than the scheme we ensure that the interior fluxes are cancelled in a telescopic sum, with the result that the correct conservation law is preserved by the numerical scheme.

1.15 The Engquist-Osher Upwind Scheme

An alternative construction of a first order accurate upwind scheme was proposed by Engquist and Osher (**?**). Assuming that the flux function $f(u)$ is convex, with a minimum at u^*, it can be split as

$$f(u) = f^+(u) + f^-(u) \tag{1.81}$$

where

$$f^+(u) = \begin{cases} f(u), & u \geq u^* \\ f(u^*), & u \leq u^* \end{cases} \tag{1.82}$$

and

$$f^-(u) = \begin{cases} f(u), & u \leq u^* \\ f(u^*), & u \geq u^* \end{cases} \tag{1.83}$$

In the case of Burger's equation

$$f(u) = \frac{u^2}{2}$$

$$f^+(u) = \begin{cases} \frac{u^2}{2}, & u \geq 0 \\ 0, & u \leq 0 \end{cases}$$

$$f^-(u) = \begin{cases} \frac{u^2}{2}, & u \leq 0 \\ 0, & u \geq 0 \end{cases}$$

Now the numerical flux is defined as

$$h_{j+\frac{1}{2}} = f^+(v_j) + f^-(v_{j+1})$$

In zones where the wave speed has a constant sign this is identical to the upwind scheme defined in section 1.12, but it is different in transition zones where the wave speed changes sign.

Consider, for example, the case of Burger's equation with $v_j < 0, v_{j+1} > 0$. The Engquist-Osher flux is

$$h^{EO}_{j+\frac{1}{2}} = f^+_j + f^-_{j+1} = 0$$

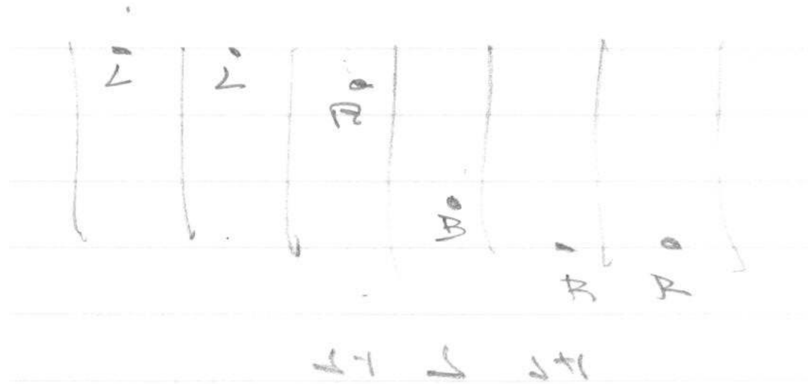

Figure 1.9: Shock structure of the Engquist-Osher scheme

while the upwind flux is

$$h^u_{j+\frac{1}{2}} = \begin{cases} f_j & if \ v_{j+1} + v_j > 0 \\ \frac{1}{2}(f_j + f_{j+1}) & if \ v_{j+1} + v_j = 0 \\ f_{j+1} & if \ v_{j+1} + v_j < 0 \end{cases}$$

In the case of a shock there may be a transition from left state v_j with $a(v_j) > 0$ to a right state v_{j+1} with $a(v_{j+1}) < 0$, so that

$$h^{EO}_{j+\frac{1}{2}} = f^+_j + f^-_{j+1} = f_j + f_{j+1}$$

Consider again Burger's equation where we now examine the numerical structure of a stationary shock with left and right states $v_L > 0$ and $v_R < 0$.
It now turns out that a structure with two interior points is possible, as illustrated in Figure 1.9. Here

$$v_L \geq v_A \geq 0, \ \ 0 \geq v_B \geq v_R$$

and

$$\begin{aligned} h_{j-\frac{1}{2}} &= f_L \\ h_{j+\frac{1}{2}} &= f^+(v_A) + f^-(v_B) = f_A + f_B \\ h_{j+\frac{3}{2}} &= f_R \end{aligned}$$

Thus every point is in equilibrium if

$$f_A + f_B = f_L = f_R$$

or

$$v_A^2 + v_B^2 = v_L^2 = v_R^2$$

Hence, given v_A

$$v_B = -\sqrt{v_L^2 - v_A^2}$$

1.16 The Jameson-Schmidt-Turkel Scheme

According to section 1.9 an LED scheme with fixed coefficients is limited to first order accuracy. We now examine ways of constructing switched higher order schemes which revert to first order accuracy only in the vicinity of extrema. One of the simplest such formulations is the Jameson Schmidt Turkel (JST) scheme [(**?**)].

This scheme blends low and high order diffusion using a switch that eliminates the high order diffusion at extrema. Suppose that the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.84}$$

is approximated by the semi-discrete finite volume scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0 \tag{1.85}$$

In the JST scheme the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - d_{j+\frac{1}{2}} \tag{1.86}$$

where the diffusive flux has the form

$$d_{j+\frac{1}{2}} = \epsilon^{(2)}_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - \epsilon^{(4)}_{j+\frac{1}{2}} (\Delta v_{j+\frac{3}{2}} - 2\Delta v_{j+\frac{1}{2}} + \Delta v_{j-\frac{1}{2}}) \tag{1.87}$$

with

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

Let $a_{j+\frac{1}{2}}$ be the numerically estimated wave speed

$$a_{j+\frac{1}{2}} = \begin{cases} \frac{f_{j+1}-f_j}{v_{j+1}-v_j}, & v_{j+1} \neq v_j \\ \frac{\partial f}{\partial u}|_{u=v_j}, & v_{j+1} = v_j \end{cases}$$

as before. Then we have the following.
<u>*Theorem*</u>: The JST scheme is LED if whenever v_j or v_{j+1} is an extremum

$$\epsilon^{(2)}_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|, \epsilon^{(4)}_{j+\frac{1}{2}} = 0 \tag{1.88}$$

<u>*Proof*</u>: Suppose v_j is an extremum. Then the second condition ensures that

$$\epsilon^{(4)}_{j+\frac{1}{2}} = 0, \epsilon^{(4)}_{j-\frac{1}{2}} = 0$$

Hence the scheme reduces to the 3 point scheme

$$\Delta x \frac{dv_j}{dt} = (\epsilon^{(2)}_{j+\frac{1}{2}} - \frac{1}{2}\alpha_{j+\frac{1}{2}})\Delta v_{j+\frac{1}{2}} - (\epsilon^{(2)}_{j-\frac{1}{2}} + \frac{1}{2}\alpha_{j-\frac{1}{2}})\Delta v_{j-\frac{1}{2}}$$

and according to the first condition the coefficients of both $v_{j+1} - v_j$ and $v_{j-1} - v_j$ are non-negative, satisfying the requirements for an LED scheme.

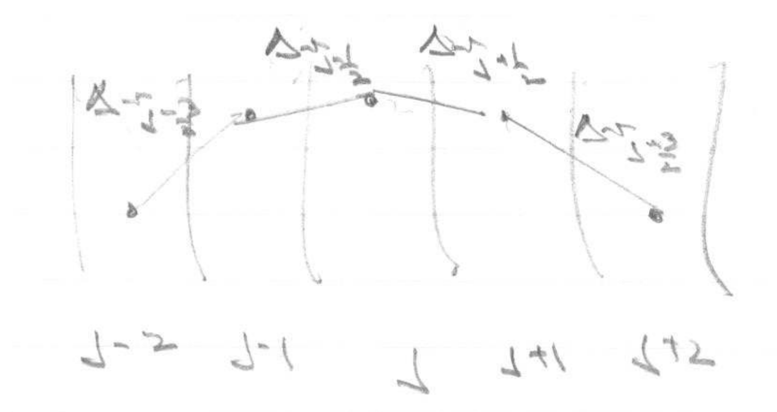

Figure 1.10: JST scheme at an extremum. $\Delta v_{j+\frac{3}{2}}$ and $\Delta v_{j-\frac{1}{2}}$ have opposite signs. Also $\Delta v_{j+\frac{1}{2}}$ and $\Delta v_{j-\frac{3}{2}}$ have oppsite signs. Thus $\epsilon^{(4)}_{j+\frac{1}{2}} = 0$ and $\epsilon^{(4)}_{j-\frac{1}{2}} = 0$

In order to construct coefficients $\epsilon^{(2)}_{j+\frac{1}{2}}$ and $\epsilon^{(4)}_{j+\frac{1}{2}}$ satisfying conditions (1.88) define the function

$$R(u, v) = \left| \frac{u - v}{|u| + |v|} \right|^q$$

where $q \geq 1$. Then if u and v have opposite signs

$$R(u, v) = 1$$

Now set

$$\epsilon^{(2)}_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} Q_{j+\frac{1}{2}}$$

and

$$\epsilon^{(4)}_{j+\frac{1}{2}} = \beta_{j+\frac{1}{2}} (1 - Q_{j+\frac{1}{2}})$$

where

$$Q_{j+\frac{1}{2}} = R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

Since $\Delta v_{j+\frac{3}{2}}$ and $\Delta v_{j-\frac{1}{2}}$ have opposite signs if either v_j or v_{j+1} is an extremum, the scheme will be LED if

$$\alpha_{j+\frac{1}{2}} \geq \frac{1}{2} |a_{j+\frac{1}{2}}|$$

Typically

$$\beta_{j+\frac{1}{2}} = K |a_{j+\frac{1}{2}}|$$

where in the case of steady state calculations K can be tuned to maximize the rate of convergence to a steady state.

1.17 Essentially Local Extremum Diminishing (ELED) Schemes

The JST scheme as presented in section 1.16 has the disadvantage that it reverts to first order accuracy at smooth extrema. In order to circumvent this loss of accuracy we can relax slightly the requirements of an LED scheme by introducing the concept of an essentially local extremum diminishing (ELED) scheme, defined as an scheme which becomes LED in the limit as the mesh width $\Delta x \to 0$.

The JST scheme can be converted to an ELED scheme by redefining the switching function as

$$R(u,v) = \left| \frac{u-v}{max((|u|+|v|), \epsilon \Delta x^r)} \right|^q$$

where a threshold has been introduced in the denominator. The coefficient ϵ should have the physical dimensions of u divided by a length scale to the power r. It is shown in [(?)] that the scheme is second order accurate and ELED if

$$q \geq 2, r = \frac{3}{2}$$

1.18 Symmetric Limited Positive (SLIP) Schemes

We now discuss schemes which obtain second order accuracy by subtracting an anti-diffusive term based on neighboring values to cancel the first order diffusive term, but limit the anti diffusion in the vicinity of extrema in order to preserve the LED property. This idea was first introduced in the flux corrected transport (FCT) scheme proposed by Boris and Book [(?)]. Their formulation used two stages. The first stage consisted of a first order accurate positive scheme. The second stage added negative diffusion to cancel the first order error, subject to a limit on its magnitude at any location where it would cause the appearance of a new extremum in the solution. In this section we show how the same result can be accomplished with a single stage scheme.

We can subtract the anti-diffusion symmetrically or from the upwind side, leading to schemes which will be classified as symmetric limited positive (SLIP) and upstream limited positive (USLIP) schemes respectively. A symmetric scheme of this type was proposed by Jameson [(?)]. A short time later Helen Yee proposed a very similar scheme [(?)] which has been popularly known as a symmetric TVD scheme.

The formulation here uses the concept of limited averages [(?)]. A limited average $L(u,v)$ of u and v is defined as an average with the following properties:

- P1: $L(u,v) = L(v,u)$
- P2: $L(\alpha u, \alpha v) = \alpha L(u,v)$
- P3: $L(u,u) = u$
- P4: $L(u,v) = 0$ if u and v have opposite signs; otherwise $L(u,v)$ has the sign as u and v; or the same sign as whichever is nonzero if $u = 0$ or $v = 0$

The first three properties are satisfied by the arithmetic average. It is the fourth property that distinguishes the limited average.

Suppose now that the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

is approximated by the semi-discrete scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$

as before. Define the numerical flux as

$$h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - d_{j+\frac{1}{2}}$$

where $d_{j+\frac{1}{2}}$ is the numerical diffusive flux. Let

$$\Delta v_{j+\frac{1}{2}} = v_{j+1} - v_j$$

By subtracting $\frac{1}{2}(\Delta v_{j+\frac{3}{2}} + \Delta v_{j-\frac{1}{2}})$ from $\Delta v_{j+\frac{1}{2}}$ we could produce a diffusive term

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} \left(\Delta v_{j+\frac{1}{2}} - \frac{1}{2}(\Delta v_{j+\frac{3}{2}} + \Delta v_{j-\frac{1}{2}}) \right)$$

which approximates $\alpha \Delta x^3 \frac{\partial^3 u}{\partial x^3}$. However this would lead to a 5 point scheme which does not satisfy the LED condition.

In order to circumvent this we replace the arithmetic average of $\Delta v_{j+\frac{3}{2}}$ and $\Delta v_{j-\frac{1}{2}}$ by their limited average and define

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} \left(\Delta v_{j+\frac{1}{2}} - L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) \right)$$

Hence $\alpha_{j+\frac{1}{2}} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$ is a limited anti diffusive term, and it will be verified below that the scheme is LED if $\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|$.

We first give some examples of limited averages. Define

$$S(u,v) = \frac{1}{2}\left(sign\ (u) + sign\ (v)\right)$$

so that

$$S(u,v) = \begin{cases} 1 & \text{when } u > 0 \text{ and } v > 0 \\ 0 & \text{when u and v have opposite sign} \\ -1 & \text{when } u < 0 \text{ and } v < 0 \end{cases}$$

Some well known limited averages are

- Min mod: $S(u,v) \min(|u|, |v|)$
- Van Leer: $S(u,v) 2\frac{|u||v|}{|u|+|v|}$
- Superbee: $S(u,v) \max \left\{ \min(2|u|, |v|), \min(|u|, 2|v|) \right\}$

Limited averages can be characterized in the following manner. Define

$$\phi(r) = L(1,r) = L(r,1)$$

so that by P2, setting $\alpha = \frac{1}{u}$

$$L(1, \frac{v}{u}) = \frac{1}{u} L(u, v)$$

Hence

$$L(u, v) = \phi(\frac{v}{u})u$$

and similarly

$$L(u, v) = \phi(\frac{u}{v})v$$

It follows on setting $u = 1, v = r$ that

$$\phi(r) = r\phi(\frac{1}{r})$$

Also by P4

$$\phi(r) = 0, r < 0$$

$$\phi(r) \geq 0, r \geq 0$$

Using these properties we can now prove that the SLIP scheme is LED if for all j

$$\alpha_{j+\frac{1}{2}} \geq \frac{1}{2}|a_{j+\frac{1}{2}}|$$

Proof: Define

$$r^+ = \frac{\Delta v_{j+\frac{3}{2}}}{\Delta v_{j-\frac{1}{2}}}, r^- = \frac{\Delta v_{j-\frac{3}{2}}}{\Delta v_{j+\frac{1}{2}}}$$

Then

$$L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) = \phi(r^+)\Delta v_{j-\frac{1}{2}}$$

and

$$L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{3}{2}}) = \phi(r^-)\Delta v_{j+\frac{1}{2}}$$

Hence the semi-discrete scheme is reduced to

$$\begin{aligned}
\Delta x \frac{dv_j}{dt} =& - \alpha_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - \frac{1}{2}\alpha_{j-\frac{1}{2}} \Delta v_{j-\frac{1}{2}} \\
&+ \alpha_{j+\frac{1}{2}}(\Delta v_{j+\frac{1}{2}} - \phi(r^+)\Delta v_{j-\frac{1}{2}}) \\
&- \alpha_{j-\frac{1}{2}}(\Delta v_{j-\frac{1}{2}} - \phi(r^-)\Delta v_{j+\frac{1}{2}}) \\
=& + \{\alpha_{j+\frac{1}{2}} - \frac{1}{2}a_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}}\phi(r^-)\}\Delta v_{j+\frac{1}{2}} \\
&- \{\alpha_{j-\frac{1}{2}} + \frac{1}{2}a_{j-\frac{1}{2}} + \alpha_{j+\frac{1}{2}}\phi(r^+)\}\Delta v_{j-\frac{1}{2}}
\end{aligned}$$

Since $\phi(r^+) \geq 0$ and $\phi(r^-) \geq 0$ the coefficient of $\Delta v_{j+\frac{1}{2}}$ is non-negative and the coefficient of $\Delta v_{j-\frac{1}{2}}$ is non-positive under the stated condition. The limiters enable the SLIP scheme to be effectively represented as a 3 point scheme.

Figure 1.11 and 1.12 illustrate the behavior of the SLIP scheme for an odd even mode and a shock wave. For an odd-even mode $\Delta v_{j+\frac{3}{2}}$ and $v_{j-\frac{1}{2}}$ have the same sign, opposite to that of $v_{j+\frac{1}{2}}$, and both reinforce $v_{j+\frac{1}{2}}$ as they would in a simple diffusion using fourth differences. With

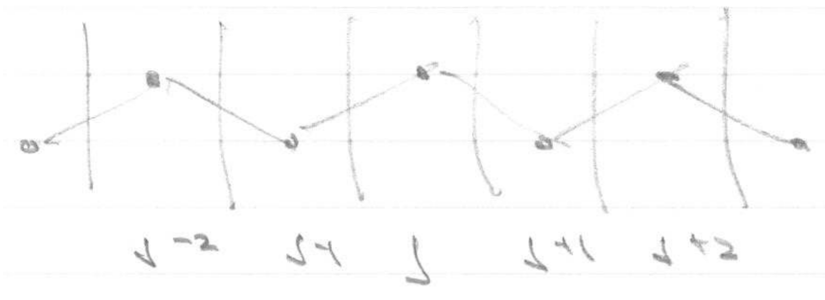

Figure 1.11: SLIP scheme for odd-even mode. $\Delta v_{j-\frac{1}{2}}$ and $\Delta v_{j+\frac{3}{2}}$ have the same sign leaving diffusive terms proportional to fourth difference in place

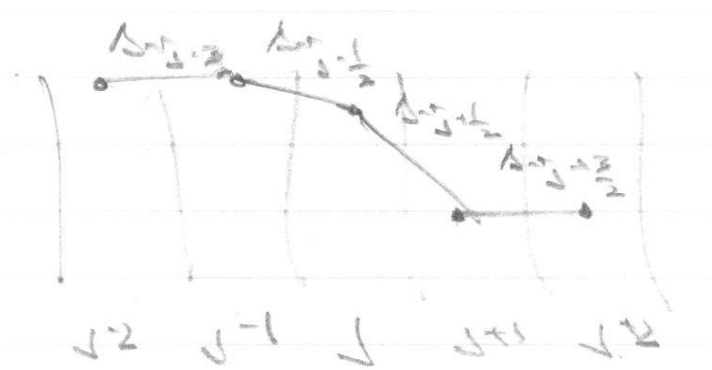

Figure 1.12: SLIP scheme at a shock. $L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) = 0, L(\Delta v_{j+\frac{5}{2}}$ and $\Delta v_{j+\frac{1}{2}}) = 0$, yielding a three point scheme at cell j.

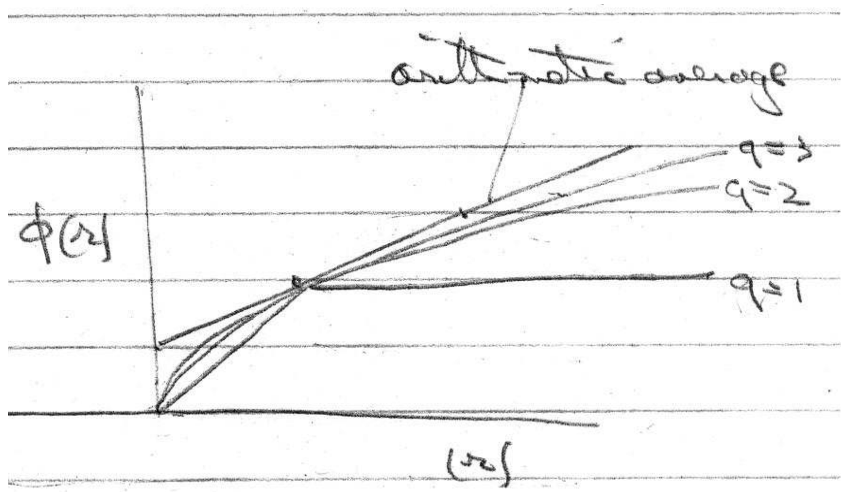

Figure 1.13: Switch ϕ as a function of r and q

$\alpha_{j+\frac{1}{2}} = \frac{1}{2}|a_{j+\frac{1}{2}}|$ the scheme allows a stationary shock with one interior point because

$$L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) = 0, L(\Delta v_{j+\frac{5}{2}}, \Delta v_{j+\frac{1}{2}}) = 0$$

with the consequence that

$$h_{j+\frac{1}{2}} = f_j = f_L \ , \ h_{j+\frac{3}{2}} = f_{j+\frac{1}{2}} = f_R$$

A general class of limiters satisfying conditions P1 - P4 can be constructed as the arithmetic average multiplied by a switch:

$$L(u,v) = \frac{1}{2}D(u,v)(u+v)$$

where $0 \leq D(u,v) \leq 1$ and $D(u,v) = 0$ if u and v have opposite signs This is realized by the formula

$$D(u,v) = 1 - \left|\frac{u-v}{u+v}\right|^q$$

where q is a positive integer. This definition contains some of the previously defined limiters as follows:

$$q = 1 \text{ gives minmod}$$

$$q = 2 \text{ gives van Leer limiters since } \frac{1}{2}(1 - \left|\frac{u-v}{u+v}\right|^2)(u+v) = \frac{2uv}{u+v}$$

As $q \to \infty, L(u,v)$ approaches a limit set by the arithmetic mean if u and v have the same sign and zero if they have opposite signs. The corresponding switch $\phi(r) = L(1,r)$ is illustrated in Figure 1.13.

With this class of limiters the SLIP scheme actually recovers a variant of the JST scheme since

$$D(u,v) = 1 - R(u,v)$$

where $R(u,v)$ is the switch used in the JST scheme. Set

$$Q_{j+\frac{1}{2}} = R(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

Then the SLIP scheme can be written as

$$\begin{aligned} d_{j+\frac{1}{2}} &= \alpha_{j+\frac{1}{2}} \left\{ \Delta v_{j+\frac{1}{2}} - \frac{1}{2}(1 - Q_{j+\frac{1}{2}})(\Delta v_{j+\frac{3}{2}} + \Delta v_{j-\frac{1}{2}}) \right\} \\ &= \alpha_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta v_{j+\frac{1}{2}} - \frac{1}{2}\alpha_{j+\frac{1}{2}}(1 - Q_{j+\frac{1}{2}})(\Delta v_{j+\frac{3}{2}} - 2\Delta v_{j+\frac{1}{2}} + \Delta v_{j-\frac{1}{2}}) \end{aligned}$$

This is the JST scheme with $K = \frac{1}{2}$.

1.19 Upstream Limited Positive (USLIP) Schemes

By adding the anti-diffusive correction purely from the upstream side one may derive a family of upstream limited positive (USLIP) schemes. Corresponding to the SLIP scheme, a USLIP scheme is obtained by setting

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} \left\{ \Delta v_{j+\frac{1}{2}} - L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) \right\}$$

if $a_{j+\frac{1}{2}} > 0$ and

$$d_{j+\frac{1}{2}} = \alpha_{j+\frac{1}{2}} \left\{ \Delta v_{j+\frac{1}{2}} - L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j+\frac{3}{2}}) \right\}$$

if $a_{j+\frac{1}{2}} < 0$. Now an analysis similar to that in section 18 reveals that if $a_{j+\frac{1}{2}} > 0$ and $a_{j-\frac{1}{2}} > 0$ while $\alpha_{j+\frac{1}{2}} = \frac{1}{2}a_{j+\frac{1}{2}}$ the scheme reduces to

$$\Delta x \frac{dv_j}{dt} = -\left\{ \frac{1}{2}\phi(r^+)a_{j+\frac{1}{2}} + (1 - \frac{1}{2}\phi(r^-))a_{j-\frac{1}{2}} \right\} \Delta v_{j-\frac{1}{2}}$$

where

$$r^+ = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}, r^- = \frac{\Delta v_{j-\frac{3}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

Thus the coefficient of $v_{j-1} - v_j$ is nonnegative and the scheme is LED if $\phi(r)$ satisfies the additional constraint $\phi(r) < 2$

1.20 Reconstruction

An alternative approach to constructing a high resolution scheme is reconstruction. This was first introduced by van Leer in the Monotone Upstream-centered Schemes for Conservation Laws (MUSCL) (?). It can be described as follows. Suppose the interface flux $h_{j+\frac{1}{2}}$ of a first order LED scheme is constructed as a function of v_j and v_{j+1}. Instead we evaluate $h_{j+\frac{1}{2}}$ from values v_L and v_R which represent estimates of the solution at the left and right side of the interface which take account of the gradient of the solution in cells j and j+1. However, in order to prevent the formation of a new extremum or the growth of an existing extremum, these values are limited based on a comparison with the gradients in neighboring cells.

In order to illustrate the process consider the case of linear advection

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

for a right running wave, $a > 0$. Suppose this is approximated by the semi-discrete scheme

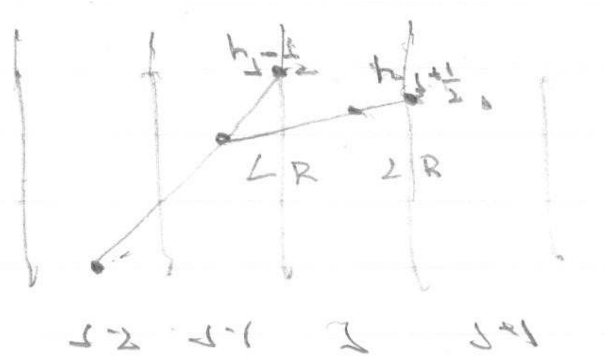

Figure 1.14: Reconstruction for linear advection

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0$$

Using an upwind flux in a first order scheme

$$h_{j+\frac{1}{2}} = av_j, h_{j-\frac{1}{2}} = av_{j-1}$$

and

$$\Delta x \frac{dv_j}{dt} = -a(v_j - v_{j-1})$$

which satisfies the conditions for an LED scheme. Suppose that we have an estimate v'_j for the slope in cell j. Then in order to improve the accuracy we can set

$$h_{j+\frac{1}{2}} = av_L$$

where

$$v_L = v_j + \frac{1}{2}\Delta x v'_j$$

Note that since this is a finite volume scheme v_j represents the average value of v in cell j. But with a linear variation of v in the cell this is also the point value at cell center. With higher order reconstruction using polynomials of degree > 1 to represent the solution, it becomes necessary to distinguish between the point value at the cell center and the average value in the cell, since these are no longer the same. Here consistent with the use of an upwind flux, we can estimate v'_j from the upwind side as

$$v'_j = \frac{v_j - v_{j-1}}{\Delta x} = \frac{\Delta v_{j-\frac{1}{2}}}{\Delta x}$$

so that

$$h_{j+\frac{1}{2}} = a(v_j + \frac{1}{2}\Delta v_{j-\frac{1}{2}})$$

$$h_{j-\frac{1}{2}} = a(v_{j-1} + \frac{1}{2}\Delta v_{j-\frac{3}{2}})$$

and

$$\Delta x \frac{dv_j}{dt} = -\frac{1}{2}a(3v_j - 4v_{j-1} + v_{j-2}) = -2a(v_j - v_{j-1}) + \frac{1}{2}a(v_j - v_{j-2})$$

This is a second order accurate estimate of $\frac{\partial v}{\partial x}$, but it violates the conditions for an LED scheme because the coefficient of $v_{j-2} - v_j$ is negative. In order to remedy this we can estimate the slope as

$$v'_j = \frac{1}{\Delta x} L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}})$$

where L is a limited average satisfying properties P1-P4 in section 1.18. Now define

$$r^+ = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}, r^- = \frac{\Delta v_{j-\frac{3}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

Then, using the notation of section 16,

$$h_{j+\frac{1}{2}} = a(v_j + \frac{1}{2}\phi(r^+)(v_j - v_{j-1}))$$

$$h_{j-\frac{1}{2}} = a(v_{j-1} + \frac{1}{2}\phi(r^-)(v_j - v_{j-1}))$$

and

$$\Delta x \frac{dv_j}{dt} = -a(1 + \frac{1}{2}\phi(r^+) - \frac{1}{2}\phi(r^-))(v_j - v_{j-1})$$

Now to ensure that the coefficient of $v_{j-1} - v_j$ is non-negative, we must also require that

$$\phi(r^+) - \phi(r^-) \geq -2 \tag{1.89}$$

If $\phi(r^+) \geq 0$ with $\phi(r) = 0$ when $r \leq 0$ this is satisfied if $\phi(r)$ satisfies the additional requirement that

$$\phi(r) \leq 2, r > 0$$

It was pointed out by Spekrizse (**?**) that actually (1.89) is satisfied if for all r

$$\phi(r) \geq= -\alpha \ , \ \phi(r) \leq 2 - \alpha$$

where $0 \leq \alpha \leq 2$. Thus the class of admissible limiters can be expanded to include limiters which are negative for some range of r.

For second order accuracy v'_j should approach $\frac{\Delta v_{j-\frac{1}{2}}}{\Delta x}$ as $\Delta v_{j+\frac{1}{2}} \rightarrow \Delta v_{j-\frac{1}{2}}$, thus recovering the second order accurate upwind scheme. This implies

$$\phi(1) = 1$$

consistent with property P3. These constraints are satisfied by the minmod, van Leer and superbee limiters with the latter reading the limits $\phi(r) \leq 2r$ in the interval $0 < r \leq \frac{1}{2}$ and $\phi(r) \leq 2$ in the interval $2 \leq r \leq \infty$. Notice also that when v_j is a maximum, as illustrated in Figure 1.14, it is $v_{L,j-\frac{1}{2}}$ which becomes too large and causes $\frac{dv_j}{dt}$ to become positive in the absence of limiters.

1.21 Reconstruction for a Nonlinear Conservation Law

We consider now the case of the nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.90}$$

approximated by the semi-discrete scheme

$$\Delta x \frac{dv_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0 \tag{1.91}$$

The solution may now contain waves traveling in either direction. Using the Engquist-Osher splitting we now construct the numerical flux as

$$h_{j+\frac{1}{2}} = f^+(v_{L,j+\frac{1}{2}}) + f^-(v_{R,j+\frac{1}{2}}) \tag{1.92}$$

where

$$f(u) = f^+(u) + f^-(u) \tag{1.93}$$

and

$$\frac{\partial f^+}{\partial u} = a^+(u) \geq 0, \frac{\partial f^-}{\partial u} = a^-(u) \leq 0 \tag{1.94}$$

while

$$v_{L,j+\frac{1}{2}} = v_j + \frac{1}{2} L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) \tag{1.95}$$

and

$$v_{R,j+\frac{1}{2}} = v_{j+1} - \frac{1}{2} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j+\frac{1}{2}}) \tag{1.96}$$

Thus

$$\Delta x \frac{dv_j}{dt} = -\Big(f^-(v_{R,j+\frac{1}{2}}) - f^-(v_{R,j+\frac{1}{2}})\Big) - \Big(f^+(v_{L,j+\frac{1}{2}}) - f^+(v_{L,j+\frac{1}{2}})\Big) \tag{1.97}$$

Now by the mean value theorem

$$\frac{f^-(v_{R,j+\frac{1}{2}}) - f^-(v_{R,j-\frac{1}{2}})}{v_{R,j+\frac{1}{2}} - v_{R,j-\frac{1}{2}}} = a^-(v_R^*) \leq 0 \tag{1.98}$$

and

$$\frac{f^+(v_{L,j+\frac{1}{2}}) - f^+(v_{L,j-\frac{1}{2}})}{v_{L,j+\frac{1}{2}} - v_{L,j-\frac{1}{2}}} = a^+(v_L^*) \geq 0 \tag{1.99}$$

where v_R lies in the range between $v_{R,j+\frac{1}{2}}$ and $v_{R,j-\frac{1}{2}}$ and v_L in the range between $v_{L,j+\frac{1}{2}}$ and ${}_{L,j-\frac{1}{2}}$. Denote the slope ratio as

$$r_j = \frac{\Delta v_{j+\frac{1}{2}}}{\Delta v_{j-\frac{1}{2}}}$$

Then

$$v_{R,j+\frac{1}{2}} - v_{R,j-\frac{1}{2}} = v_{j+1} - \frac{1}{2} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j+\frac{1}{2}}) - v_j + \frac{1}{2} L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) = (1 - \frac{1}{2}\phi(r_{j+1}) + \frac{1}{2}\phi(\frac{1}{r_j}))\Delta v_{j+\frac{1}{2}}$$

and

$$v_{L,j+\frac{1}{2}} - v_{L,j-\frac{1}{2}} = v_j + \frac{1}{2} L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) - v_{j-1} - \frac{1}{2} L(\Delta v_{j-\frac{1}{2}}, \Delta v_{j-\frac{3}{2}}) = (1 + \frac{1}{2}\phi(r_j) - \frac{1}{2}\phi(\frac{1}{r_{j-1}}))\Delta v_{j-\frac{1}{2}}$$

Thus equation 1.98 can be rearranged as

$$\Delta x \frac{dv_j}{dt} = -a^-(v_R^*)(1 - \frac{1}{2}\phi(r_{j+1}) + \frac{1}{2}\phi(\frac{1}{r_j}))\Delta v_{j+\frac{1}{2}} - a^+(v_L^*)(1 + \frac{1}{2}\phi(r_j) - \frac{1}{2}\phi(\frac{1}{r_{j-1}}))\Delta v_{j-\frac{1}{2}} \tag{1.100}$$

The coefficients of $v_{j+1} - v_j$ and $v_{j-1} - v_j$ are both non-negative if for all r and s

$$\phi(r) + \phi(s) \geq -2$$

and accordingly the scheme is LED if the limiter satisfies

$$\phi(r) \geq -\alpha, \phi(r) \leq 2 - \alpha \tag{1.101}$$

where $0 \leq \alpha \leq 2$, as in the linear case. If we use a limiter satisfying

$$\phi(r) = 0, r \leq 0$$

$$0 \leq \phi(r) \leq 2, r > 0$$

then if v_j is a maximum or a minimum $\Delta v_{j+\frac{1}{2}}$ and $\Delta v_{j-\frac{1}{2}}$ have opposite sings so that

$$L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) = 0$$

and

$$v_{L,j+\frac{1}{2}} = v_j$$

If both

$$and$$

are positive

$$L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) \leq 2\Delta v_{j+\frac{1}{2}}$$

and

$$v_{L,j+\frac{1}{2}} \leq v_{j+1}$$

while if both $\Delta v_{j-\frac{1}{2}}$ and $\Delta v_{j+\frac{1}{2}}$ are negative

$$L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{1}{2}}) \geq -2\Delta v_{j+\frac{1}{2}}$$

and

$$v_{L,j+\frac{1}{2}} \geq v_{j+1}$$

Thus with the class of limiter $v_{L,j+\frac{1}{2}}$ and similarly $v_{R,j+\frac{1}{2}}$ lie in the range between v_j and v_{j+1}

$$\min(v_j, v_{j+1}) \leq v_{L,j+\frac{1}{2}} \leq \max(v_j, v_{j+1})$$

and

$$\min(v_j, v_{j+1}) \leq v_{R,j-\frac{1}{2}} \leq \max(v_j, v_{j+1})$$

Consider also the fully discrete scheme

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}) \tag{1.102}$$

where

$$h_{j+\frac{1}{2}} = f^-(v^n_{R,j+\frac{1}{2}}) + f^+(v^n_{L,j+\frac{1}{2}}) \tag{1.103}$$

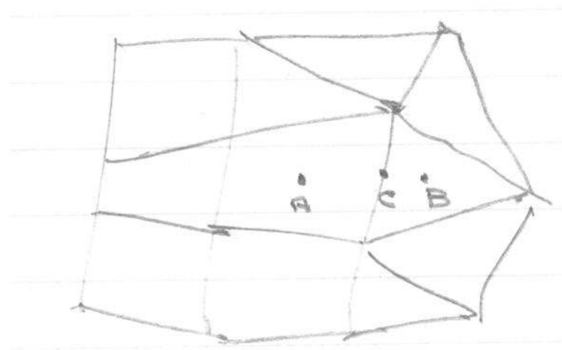

Figure 1.15: Reconstruction for an unstructured mesh

Following the same argument

$$v_j^{n+1} = A^-_{j+\frac{1}{2}} \frac{\Delta t}{\Delta x} v^n_{j+1} + (1 - \frac{\Delta t}{\Delta x}(A^-_{j+\frac{1}{2}} + A^+_{j-\frac{1}{2}}))v^n_j + A^+_{j-\frac{1}{2}} \frac{\Delta t}{\Delta x} v^n_{j-1} \tag{1.104}$$

where

$$A^-_{j+\frac{1}{2}} = -a^-(v^*_R)(1 - \frac{1}{2}\phi(r_{j+1}) + \frac{1}{2}\phi(\frac{1}{r_j}))$$

and

$$A^+_{j-\frac{1}{2}} = a^+(v^*_L)(1 + \frac{1}{2}\phi(r_j) - \frac{1}{2}\phi(\frac{1}{r_{j-1}}))$$

Under condition 1.102

$$A^-_{j+\frac{1}{2}} \geq 0, A^+_{j-\frac{1}{2}} \geq 0.$$

Also they are both bounded so we can choose Δt small enough that the coefficient of v^n_j is non-negative, and the scheme is LED.

The reconstruction approach proves to be particularly useful in the construction of high resolution schemes on unstructured meshes for multi-dimensional problems. Then one can use an estimate of the gradient of the solution in each cell to recover left and right values at the center of the face separating any two cells as illustrated in Figure 1.15. For example at the edge center C

$$v_L = v_A + \nabla v_A \cdot (\overrightarrow{x_C} - \overrightarrow{x_A})$$

$$v_R = v_B + \nabla v_B \cdot (\overrightarrow{x_C} - \overrightarrow{x_B})$$

where ∇v_A and ∇v_B are appropriately limited estimates of the gradient in cells A and B.

1.22 SLIP Reconstruction

An alternative reconstruction formula can be derived from the SLIP scheme by setting

$$v_{L,j+\frac{1}{2}} = v_j + \frac{1}{2} L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j+\frac{1}{2}}) \tag{1.105}$$

$$v_{R,j+\frac{1}{2}} = v_{j+1} - \frac{1}{2}L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) \tag{1.106}$$

so that the difference

$$v_{R,j+\frac{1}{2}} - v_{L,j+\frac{1}{2}} = \Delta v_{j+\frac{1}{2}} - L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}})$$

is the same as that used in the construction of the artificial diffusion in the SLIP scheme, while the same slope estimate centered at $j + \frac{1}{2}$ is used to calculate both $v_{L,j+\frac{1}{2}}$ and $v_{R,j+\frac{1}{2}}$.

Defining the slope ratios

$$r^+ = \frac{\Delta v_{j+\frac{3}{2}}}{\Delta v_{j+\frac{1}{2}}}, r^- = \frac{\Delta v_{j-\frac{3}{2}}}{\Delta v_{j-\frac{1}{2}}} \tag{1.107}$$

we now find that

$$\begin{aligned} v_{R,j+\frac{1}{2}} - v_{R,j-\frac{1}{2}} &= v_{j+1} - \frac{1}{2}L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) - v_j + \frac{1}{2}L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{3}{2}}) \\ &= \Delta v_{j+\frac{1}{2}} - \frac{1}{2}\phi(r^+)\Delta v_{j-\frac{1}{2}} + \frac{1}{2}\phi(r^-)\Delta v_{j+\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} v_{L,j+\frac{1}{2}} - v_{L,j-\frac{1}{2}} &= v_j + \frac{1}{2}L(\Delta v_{j+\frac{3}{2}}, \Delta v_{j-\frac{1}{2}}) - v_{j-1} - \frac{1}{2}L(\Delta v_{j+\frac{1}{2}}, \Delta v_{j-\frac{3}{2}}) \\ &= \Delta v_{j-\frac{1}{2}} + \frac{1}{2}\phi(r^+)\Delta v_{j-\frac{1}{2}} - \frac{1}{2}\phi(r^-)\Delta v_{j+\frac{1}{2}} \end{aligned}$$

Thus, following the same argument that was used in the previous section, the semi-discrete scheme defined by equations (1.91) - (1.96), can be rearranged as

$$\Delta x \frac{dv_j}{dt} = -a^-(v_R^*)(1+\frac{1}{2}\phi(r^-)+\frac{1}{2}a^+(v_L^*)\phi(r^-)\Delta v_{j+\frac{1}{2}} - a^+(v_L^*)(1+\frac{1}{2}\phi(r^+)-\frac{1}{2}a^-(v_L^*)\phi(r^+))\Delta v_{j-\frac{1}{2}} \tag{1.108}$$

The coefficients of $v_{j+1} - v_j$ and $v_{j-1} - v_j$ are both nonnegative if for all r

$$\phi(r) \geq 0$$

and correspondingly the semi-discrete scheme is LED.

The fully discrete scheme described by equations (1.102) and (1.103) can be expressed in the form (1.104) where now

$$A^-_{j+\frac{1}{2}} = -a^-(v_R^*)(1 + \frac{1}{2}\phi(r^-)) + \frac{1}{2}a^+(v_L^*)\phi(r^-)$$

and

$$A^+_{j-\frac{1}{2}} = a^+(v_L^*)(1 + \frac{1}{2}\phi(r^+)) - \frac{1}{2}a^-(v_R^*)\phi(r^+)$$

Again, provided that $\phi(r)$ is nonnegative

$$A^-_{j+\frac{1}{2}} \geq 0, A^+_{j-\frac{1}{2}} \geq 0$$

and if $\phi(r)$ is also bounded

$$0 \leq \phi(r) \leq M \tag{1.109}$$

we can choose Δt small enough that the coefficient of v_j^n is positive, and the scheme is LED.

The SLIP reconstruction requires a less restrictive bound on the maximum value of $\phi(r)$ than the upwind reconstruction defined by equations (1.95) and (1.96). This has the advantage that enables the use of a limiter that is closer to the arithmetic average. In particular one may take

$$\phi(r) = 0, r \leq 0$$

$$\phi(r) = \frac{1}{2}(1+r)(1 - \left|\frac{1-r}{1+r}\right|^q), r > 0$$

for an arbitrary value of q. This approaches the arithmetic average from below more closely as q is increased, while for large values of r

$$\phi(r) \leq q$$