AERO-STRUCTURAL WING DESIGN OPTIMIZATION USING HIGH-FIDELITY SENSITIVITY ANALYSIS

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Outline

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  – The need for aero-structural sensitivities
  – Sensitivity analysis methods

• Theory
  – Adjoint sensitivity equations
  – Lagged aero-structural adjoint equations
  – Complex-step derivative approximation

• Results
  – Aero-structural analysis and sensitivities
  – Wing optimization

• Conclusions
High-Fidelity Wing Design Optimization

- Start from a conceptual design with a baseline geometry.
- Used for preliminary design of the 3D shape of the wing.
- For transonic configurations shocks are present and must be analyzed using CFD.
- High-fidelity analysis needs high-fidelity parameterization, e.g. to smooth shocks.
- Gradient-based optimization is the most efficient.
Aero-Structural Wing Design Optimization

• Aerodynamics traditionally has used a shape corresponding to the flying shape of the wing, assuming that shape can be reproduced.

• Wing shape depends on aerodynamic solution, so need to couple aerodynamic and structural analyses to obtain the true solution, specially for unusual designs.

• Want to optimize the structure as well, since there is a trade-off between aerodynamic performance and structural weight:

\[ \text{Range} \propto \frac{L}{D} \ln \left( \frac{W_i}{W_f} \right) \]
The Need for Aero-Structural Sensitivities

- Sequential optimization does not lead to the true optimum.
- Aero-structural optimization requires coupled sensitivities.
- Add structural element sizes to the design variables.
- Including structures in the high-fidelity wing optimization will allow larger changes in the design.
Methods for Sensitivity Analysis

• **Finite-Difference:** very popular; easy, but lacks robustness and accuracy; run solver $n$ times.

$$f'(x) \approx \frac{f(x + h) - f(x)}{h} + O(h)$$

• **Complex-Step Method:** relatively new; accurate and robust; easy to implement and maintain; run solver $n$ times.

$$f'(x) \approx \frac{\text{Im} [f(x + ih)]}{h} + O(h^2)$$

• **Algorithmic/Automatic/Computational Differentiation:** accurate; ease of implementation and cost varies.

• **(Semi)-Analytic Methods:** efficient and accurate; long development time; cost can be independent of $n$. 
Objective Function and Governing Equations

Want to minimize scalar objective function,

\[ I = I(x, y), \]

which depends on:
- \( x \): vector of design variables, e.g. structural plate thickness.
- \( y \): state vector, e.g. structural displacements.

Physical system is modeled by a set of governing equations:

\[ R(x, y(x)) = 0, \]

where:
- Same number of state and governing equations, \( n_R \)
- \( n_x \) design variables.
Variational Equations

Total variation of the objective function:

\[ \delta I = \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y. \]

Variation of the governing equations,

\[ \delta R = \frac{\partial R}{\partial x} \delta x + \frac{\partial R}{\partial y} \delta y = 0. \]
Adjoint Sensitivity Equations

Since the variation of the governing equations must be zero, we can add it to the total variation of the objective,

\[
\delta I = \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \psi^T \left( \frac{\partial R}{\partial x} \delta x + \frac{\partial R}{\partial y} \delta y \right),
\]

where \( \psi \) is a vector of arbitrary components known as *adjoint variables*. 

Re-arrange terms,

\[
\delta I = \left( \frac{\partial I}{\partial x} + \psi^T \frac{\partial R}{\partial x} \right) \delta x + \left( \frac{\partial I}{\partial y} + \psi^T \frac{\partial R}{\partial y} \right) \delta y.
\]

If term in blue were zero, term in red would represent the total variation of the objective with respect to the design variables.
Adjoint Sensitivity Equations

Since the adjoint variables are arbitrary, we can find a set such that,

\[
\left[ \frac{\partial R}{\partial y} \right]^T \psi (n_R \times n_R) = - \frac{\partial I^T}{\partial y} (n_R \times 1)
\]

Adjoint valid for all design variables.

Now the total sensitivity of the objective is:

\[
\frac{dI}{dx} (1 \times n_R) = \frac{\partial I}{\partial x} (1 \times n_x) + \psi^T (1 \times n_R) \frac{\partial R}{\partial x} (n_R \times n_x)
\]

The partial derivatives are inexpensive, since they don’t require the solution of the governing equations.
Two coupled disciplines: Aerodynamics (A) and Structures (S).

\[ R = \begin{bmatrix} R_A \\ R_S \end{bmatrix}, \quad y = \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_A \\ \psi_S \end{bmatrix} \]

Flow variables, \( w \), five for each grid point.

Structural displacements, \( u \), three for each structural node.
Aero-Structural Adjoint Equations

\[
\begin{bmatrix}
\frac{\partial R_A}{\partial w} & \frac{\partial R_A}{\partial u} \\
\frac{\partial R_S}{\partial w} & \frac{\partial R_S}{\partial u}
\end{bmatrix}^T \begin{bmatrix}
\psi_A \\
\psi_S
\end{bmatrix} = \begin{bmatrix}
\frac{\partial C_D}{\partial w} \\
\frac{\partial C_D}{\partial u}
\end{bmatrix}
\]

- \(\frac{\partial R_A}{\partial w}\): a change in one of the flow variables affects only the residuals of its cell and the neighboring ones.
- \(\frac{\partial R_A}{\partial u}\): wing deflections cause the mesh to warp, affecting the residuals.
- \(\frac{\partial R_S}{\partial w}\): since \(R_S = Ku - f\), this is equal to \(-\frac{\partial f}{\partial w}\).
- \(\frac{\partial R_S}{\partial u}\): equal to the stiffness matrix, \(K\).
- \(\frac{\partial C_D}{\partial w}\): can be obtained from the integration of pressures that computes the total \(C_D\).
- \(\frac{\partial C_D}{\partial u}\): wing displacement changes the surface boundary over which drag is integrated.
Lagged Aero-Structural Adjoint Equations

Since the factorization of the complete residual sensitivity matrix is impractical, decouple the system and lag the adjoint variables,

\[
\begin{align*}
\left[ \frac{\partial R_A}{\partial w} \right]^T \psi_A &= \frac{\partial C_D}{\partial w} - \left[ \frac{\partial R_S}{\partial w} \right]^T \tilde{\psi}_S \\
\left[ \frac{\partial R_S}{\partial u} \right]^T \psi_S &= \frac{\partial C_D}{\partial u} - \left[ \frac{\partial R_A}{\partial u} \right]^T \tilde{\psi}_A.
\end{align*}
\]

Aerodynamic adjoint equations

Structural adjoint equations

Lagged adjoint equations are the single discipline ones with an added forcing term that takes the coupling into account.

System is solved iteratively, much like the aero-structural analysis.
Total Sensitivity

The aero-structural sensitivities of the drag coefficient with respect to wing shape perturbations are,

\[
\frac{dC_D}{dx} = \frac{\partial C_D}{\partial x} - \psi_T^T \frac{\partial R_A}{\partial x} - \psi_T^S \frac{\partial R_S}{\partial x}.
\]

• \(\partial C_D/\partial x\): a change in the geometry will change the boundary over which the pressures are integrated.

• \(\partial R_A/\partial x\): the shape perturbations affect the grid, which in turn changes the residuals.

• \(\partial R_S/\partial x\): shape perturbations only affect the stiffness matrix, so this is \(\partial K/\partial x \cdot u\).
Complex-Step Derivative Approximation

Complex variable defined as $z = x + iy$, where $x$, $y$ are real, $i = \sqrt{-1}$.

Complex function defined as $f(z) = u(z) + iv(z)$, assumed to be analytic.

Can also be derived from a Taylor series expansion about $x$ with a complex step $ih$:

$$f(x + ih) = f(x) + ihf'(x) - h^2\frac{f''(x)}{2!} - ih^{3}\frac{f'''(x)}{3!} + \ldots$$

$$\Rightarrow f'(x) = \frac{\text{Im} [f(x + ih)]}{h} + h^2\frac{f'''(x)}{3!} + \ldots$$

$$\Rightarrow f'(x) \approx \frac{\text{Im} [f(x + ih)]}{h}$$

No subtraction! Second order approximation.
Simple Numerical Example

Estimate derivative at $x = 1.5$ of the function,

$$f(x) = \frac{e^x}{\sqrt{\sin^3 x + \cos^3 x}}$$

Relative error defined as:

$$\varepsilon = \frac{|f' - f'_{\text{ref}}|}{|f'_{\text{ref}}|}$$
3D Aero-Structural Design Optimization Framework

- Aerodynamics: SYN107-MB, a parallel, multiblock Navier-Stokes flow solver.
- Structures: detailed finite element model with plates and trusses.
- Coupling: high-fidelity, consistent and conservative.
- Geometry: centralized database for exchanges (jig shape, pressure distributions, displacements.)
- Coupled-adjoint sensitivity analysis
W25 WING ANALYSIS
MACH = 0.820 , CL = 0.352

Solution 1
Upper-Surface Isobars
(Contours at 0.05 Cp)

SYMBOL  SOURCE  ALPHA  CD
---  ------  ------  ------
  AERO-STRUCTURAL      0.526  0.01186
  RIGID                  -0.039   0.01119
Structural Deflections

![Diagram showing structural deflections in x, y, and z axes. The image includes two graphs: one illustrating the deflections with red and blue lines labeled 'Aeroelastic' and 'Rigid', and another showing the x and z axes with a red and blue line respectively.](image-url)
Sensitivity Results: Spanwise Bumps

- Upper Surface
- Lower Surface

Shapes and derivatives are shown for different methods:
- Coupled adjoint
- Coupled complex step
- Coupled finite difference
- Aerodynamic adjoint
- Aerodynamic complex step
- Aerodynamic finite difference

Shape variable, $x_j$
Sensitivity Results: Chordwise Bumps

![Sensitivity Results: Chordwise Bumps](image-url)
Computational Cost Comparison

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- Aero-structural solution takes 25% longer.
- Finite-difference and complex step dependent on the number of design variables.
- Complex-step calculation is about 2.5 times slower, but no step size guessing.
- Adjoint method is much more efficient. This would be even more obvious for more design variables.
Aerodynamic Optimization Results

### Aerodynamic Optimization Results

**AERODYNAMIC W25 WING OPTIMIZATION**

**MACH = 0.820, CL = 0.352**

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**Solution 1**

Upper-Surface Isobars

(Contours at 0.05 Cp)

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Aero-Structural Optimization Results

AEROSTRUCTURAL W25 WING OPTIMIZATION
MACH = 0.820 , CL = 0.352

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<td>BASELINE</td>
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Solution 1
Upper-Surface Isobars
(Contours at 0.05 Cp)
Conclusions

• New method for high-fidelity aero-structural sensitivity analysis.

• Lagged-coupled adjoint was used to compute sensitivities of the drag coefficient with respect to wing shape perturbations.

• The sensitivities given by the aero-structural adjoint were accurate when compared to the reference values.

• The coupled-adjoint method was shown to be very efficient, making optimization with respect to hundreds design variables viable.
Future Work

• Add structural sizes to the set of design variables.

• Compute sensitivities of the structural stresses to add stress constraints.

• Optimized initial cruise weight for fixed range to get correct aero-structural trade-off.

• Implement multiple load cases.

• Use this design framework for unusual configurations such as a supersonic business jet.