Nonlinear Frequency Domain Methods Applied to the Euler and Navier-Stokes Equations

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Sponsor: Accelerated Strategic Computing Initiative (ASCI) Project - DOE

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The goal of ASCI is to perform numerical simulations of the unsteady flows inside an aircraft gas turbine engine.

Estimates of the amount of work required for a single component simulation are provided below (Source: Davis 2001).

<table>
<thead>
<tr>
<th>Component</th>
<th>Blade Rows</th>
<th>Grid Points (million)</th>
<th>Wheel Fraction</th>
<th>CPU Hours (million)</th>
<th>Execution Duration (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbine</td>
<td>9</td>
<td>94</td>
<td>$\frac{1}{6}$</td>
<td>3.0</td>
<td>510$^1$</td>
</tr>
<tr>
<td>Compressor</td>
<td>23</td>
<td>1</td>
<td>$\frac{1}{6}$</td>
<td>7.7</td>
<td>1300$^1$</td>
</tr>
</tbody>
</table>

$^1$ Based on 750 processors (1998 IBM SP2) working 8 hours per day.

The goal of this research is to develop faster algorithms for unsteady flow calculations. We will focus on improving the efficiency of temporal discretizations and will not address spatial operators.
The previous time estimates are based on a time accurate solver (TFLO). This research will propose a periodic solver capable of solving nonlinear equations in the frequency domain.
Outline

- Non-Linear Frequency Domain (NLFD) Solver
  - Governing Equations
  - Discretization
  - Gradient Based Variable Time Period (GBVTP) Method
  - Convergence Acceleration Issues

- Laminar Vortex Shedding
  - Strouhal Number Prediction
  - Variable vs Fixed Time Period Solutions

- Pitching Airfoil
  - Steady/Unsteady Solutions vs Experimental Data
  - NLFD vs Dual Time Stepping Codes (UFLO82)

- Conclusions
Navier-Stokes Equations

- The non-dimensional form of the governing equations.

\[
\frac{\partial}{\partial t} \int_{\Omega} W dV + \oint_{\partial \Omega} \vec{F}_c \cdot \vec{N} ds = \sqrt{\gamma} \frac{M_\infty}{Re_\infty} \oint_{\partial \Omega} \vec{F}_v \cdot \vec{N} ds
\]

\[
\begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho E
\end{bmatrix}
\begin{bmatrix}
\rho (u_i - b_i) \\
\rho u_1 (u_i - b_i) + p\delta_{i1} \\
\rho u_2 (u_i - b_i) + p\delta_{i2} \\
\rho E (u_i - b_i) + u_ip
\end{bmatrix}
\begin{bmatrix}
0 \\
\sigma_{i1} \\
\sigma_{i2} \\
u_j \sigma_{ij} + q_i
\end{bmatrix}
\]

- \( \vec{b} \) represents the velocity of the control volume faces.
Discretization

- The finite volume approximation to the governing equations:

\[
V \frac{\partial W}{\partial t} + \sum_{cv} \vec{F}_c \cdot \vec{S} - \sum_{cv} F_{ad} - \frac{\sqrt{\gamma M_\infty}}{Re_\infty} \sum_{cv} \vec{F}_v \cdot \vec{S} = 0
\]

\[
V \frac{\partial W}{\partial t} + F_c + F_d = 0
\]

\[
V \frac{\partial W}{\partial t} + R(W) = 0
\]

- Assume that the control volume may translate rigidly through space.
  Cell volume is not a function of time.
Represent $W$ and $R(W)$ by a Fourier series in time.

$$W = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{W}_k e^{ikt}$$

$$R(W) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{R}_k e^{ikt}$$

Resulting in a stationary system of equations in the frequency domain (NLFD).

$$i k V \hat{W}_k + \hat{R}_k = 0$$

A second transformation back into the temporal domain results in Hall’s harmonic balance approach.

$$S_n + R(W_n) = 0$$

$$S_n = i V \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} k \hat{W}_k e^{ikt}$$
NLFD vs Harmonic Balance

- NLFD and harmonic balance methods converge to identical answers.
- Both approaches require the same amount of work per solver iteration.
  N residual evaluations based on N instances of the solution.
  2 FFTS are needed for either approach.
- Unlike harmonic balance techniques, the NLFD method advances the unsteady residual in the frequency domain affording a separate pseudo time step for each wavenumber that includes the effect of the temporal derivative on the stability of the system.
- The NLFD method simplifies the implementation of convergence acceleration techniques like implicit residual averaging and multigrid (using spectral viscosity).
Solver Implementation

- Since we are dealing with solving a steady system of equations, we apply established methods to accelerate the convergence.
  - Multi-stage RK scheme with local time stepping
  - Implicit residual averaging
    - CFL limiters
  - Multigrid V or W Cycle
    - Coarse grid spectral viscosity

- We are dealing with real functions where the Fourier coefficients for the positive wavenumbers are equal to the complex conjugates of the Fourier coefficients for the negative wavenumbers. This eliminates the computation required to integrate the negative wave numbers forward in pseudo-time.
Unsteady Residual

- Restating the governing equation in the frequency domain.

\[ ikV\hat{W}_k + \hat{R}_k = 0 \quad k = 0 \ldots N_{\text{modes}} \]

- Add in a pseudo time derivative:

\[ V \frac{\partial \hat{W}_k}{\partial \tau} + ikV\hat{W}_k + \hat{R}_k = 0 \]

\[ \hat{I}_k \]

- This gradient will iteratively adjust the solution in the frequency domain to minimize the magnitude of the unsteady residual \( \hat{I}_k \).
The process of forming the unsteady residual is complicated by the nonlinearities included within the spatial operators.
Multi-Stage RK Scheme

- Implementation of a multi-stage RK scheme for a stationary system of equations (Jameson 83).

\[
W^m = W^0 + \Delta \tau \alpha^m \left( \frac{\partial W^m}{\partial \tau} \right)
\]

\[
\frac{\partial W^m}{\partial \tau} = F_c(W^m) + \beta^m F_d(W^m) + (1 - \beta^m) F_d^{m-1}
\]

- As applied in the frequency domain.

\[
\hat{W}_k^m = \hat{W}_k^0 + \Delta \tau_k \alpha^m \left( \frac{\partial \hat{W}_k^m}{\partial \tau} \right)
\]

\[
\frac{\partial \hat{W}_k^m}{\partial \tau} = \frac{1}{N} \sum_{n=0}^{N-1} \left( F_c(W_n^m) + \beta^m F_d(W_n^m) + (1 - \beta^m) F_d^{m-1} \right) e^{-ikt_n}
\]
Implicit BDF

- Implicit treatment of an A-stable discretization (Melson 93).

\[
- V \frac{\partial W_{n+1}^m}{\partial \tau} = V \frac{3W_{n+1}^m - 4W_n + W_{n-1}}{2\Delta t} + R(W_{n+1}^m)
\]

\[
W_{n+1}^{m+1} = W_{n+1}^0 - \frac{\alpha^m \Delta \tau}{V} \left( V \frac{3W_{n+1}^{m+1} - 4W_n + W_{n-1}}{2\Delta t} + R(W_{n+1}^m) \right)
\]
Implicit treatment of an NLFD method.

\[- V \frac{\partial \hat{W}_k^m}{\partial \tau} = i k V \hat{W}_k^m + \hat{R}_k(W^m)\]

\[\hat{W}_k^{m+1} = \hat{W}_k^0 - \frac{\alpha^m \Delta \tau_k}{V} \left( i k V \hat{W}_k^{m+1} + \hat{R}_k(W^m) \right)\]

Explicit

Implicit
Residual Averaging - Frequency

Continuous form of residual averaging for the advection equation.

\[
\left(1 - e^{\frac{\partial^2}{\partial \eta_i^2}}\right) \Delta \tau \bar{I} = -\Delta \tau \left( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x_i} \right)
\]

Discrete operators for the above equation.

\[
- e_k \Delta \tau_k \hat{I}_{k+1_i} + (1 + 2e_k) \Delta \tau_k \hat{I}_k - e_k \Delta \tau_k \hat{I}_{k-1_i} =
\]

\[
- \Delta \tau_k \left( ik \hat{u}_k + c \frac{\hat{u}_{k+1_i} - \hat{u}_{k-1_i}}{2 \Delta x_i} \right)
\]
Perform a von Neumann analysis to obtain the eigenvalue as a function of the discrete solution’s frequency.

\[
\Delta \tau_k \hat{I}_k = \frac{k \Delta \tau_k + \Lambda_k \sin(\rho)}{1 + 2e_k(1 - \cos(\rho))} \hat{u}_k
\]

\[
\Delta \tau_k \frac{\partial \hat{u}_k}{\partial \tau} = \lambda \hat{u}_k
\]

The above equation has been recast into a semi-implicit form suitable as the basis of a stability analysis.
CFL Limiter

- Resulting in a simple equation for the averaging coefficient $e$. 
  
  $e \to \infty$ as $\pi \gamma \to 1$.

  
  $$e_k = \frac{1}{4} \left( \frac{\pi^2 - (\pi \gamma)^2 + 2\pi \gamma - 1}{1 - \pi \gamma} \right)$$

  $$\Lambda_k = \frac{c \Delta \tau_k}{\Delta x} \quad \pi = \frac{\Lambda_k}{|\lambda|} \quad \pi \gamma = \frac{k \Delta \tau_k}{|\lambda|}$$

- If residual averaging and multigrid are used simultaneously then the time step is calculated as:

  $$\Delta \tau_k = \min \left( \frac{\Lambda_0 V}{c + k V}, \frac{|\lambda|}{k} \right)$$
Spectral Viscosity

- The residual on the coarse mesh is defined as:

\[- \frac{\partial \hat{W}_k^m}{\partial \tau} = A_{2h}^h \hat{I}_k(W_h) - \hat{I}_k(A_{2h}^h W^0) + \hat{I}_k(A_{2h}^h W^m)\]

- If the restricted residual is zero then the coarse grid correction will always be zero.

The ultimate driver in the multigrid cycle is the fine grid residual.

- Many steady-state multigrid codes take advantage of this by implementing third-order spatial dissipation operators on the fine grid and first order operators on the coarse grid.

- The additional dissipation improves the high frequency damping of the multigrid scheme resulting in an improved coarse grid correction.
Add temporal damping on the coarse grid equations.

\[ V \frac{\partial \hat{W}_k}{\partial \tau} + ikV \hat{W}_k + \hat{R}_k = \varepsilon V (ik)^2 \hat{W}_k \]

This will mitigate the dependence of the convergence rate of the residual on the unsteady terms without affecting the solution.
In the development of the nonlinear frequency domain method, we assume the time period of the fundamental harmonic.

A class of problems exists where the exact frequency of the phenomena described by discretized equations cannot be known in advance. This research has developed the Gradient Based Variable Time Period (GBVTP) method to solve for the time period of the fundamental harmonic as part of the iterative solution process.

The process of finding a solution to the unsteady flow equations is analogous to an optimization problem where the magnitude of the unsteady residual is minimized.

Taking a derivative of the square of the magnitude of the unsteady residual with respect to the time period will form a gradient that will allow us to search for the time period that minimizes the unsteady residual.
The normalized wavenumber $k$:

$$k = \frac{2\pi n}{T}$$

The unsteady residual can then be written as a function of the time period $T$.

$$\hat{I}_n = \frac{i2\pi n V}{T} \hat{W}_n + \hat{R}_n$$

The magnitude of the unsteady residual can be expressed as

$$\frac{1}{2} \frac{\partial |\hat{I}_n|^2}{\partial T} = \hat{I}_{nr} \frac{\partial \hat{I}_{nr}}{\partial T} + \hat{I}_{ni} \frac{\partial \hat{I}_{ni}}{\partial T}$$
\[ \frac{\partial \hat{I}_{nr}}{\partial T} = \frac{2\pi nV\hat{W}_{ni}}{T^2} \quad \frac{\partial \hat{I}_{ni}}{\partial T} = -\frac{2\pi nV\hat{W}_{nr}}{T^2} \]

- The gradient can be simplified by employing cross product notation.

\[ \frac{1}{2} \frac{\partial}{\partial T} \left| \hat{I}_n \right|^2 = \frac{2\pi nV}{T^2} |\hat{I}_n \times \hat{W}_n| \]

- By selecting a stable \( \Delta T \) the gradient information can be used to update the time period after each multigrid cycle in the solution process.

\[ T^{n+1} = T^n - \Delta T \frac{\partial \left| \hat{I}_n \right|^2}{\partial T} \]
Cylinder - Motivation

- To show that the NLFD method using a limited number of time varying modes could accurately resolve complex flow field physics such as the von Kármán vortex street.

- To demonstrate the ability of the GBVTP method to automatically determine the shedding frequency of the discrete equations.

- To show the impact of the GBVTP method on the solution in comparison to fixed time period results.
Laminar vortex shedding occurs behind a circular cylinder over a range of Reynolds numbers between 40 and 194 (Williamson).

It is important to note the cylinder end boundary conditions when selecting experimental studies for comparison.

Source: Taneda

Source: Williamson
Cylinder - Details

Nearfield - 257x129

Entropy contours

Three dimensional parametric survey:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temporal Resolution</td>
<td>1,3,5,7</td>
</tr>
<tr>
<td>Grid Resolution</td>
<td>129x65,193x81,257x129,385x161</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>60,70,80,90,100,110,120,130,140,150</td>
</tr>
</tbody>
</table>
Strouhal Predictions

193x81 grid

385x161 grid

Reynolds Number

Strouhal Number

1 Harmonic
3 Harmonic
5 Harmonic
7 Harmonic
Williamson 1988

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The GBVTP method will find the exact Strouhal number for the discrete equations. Performing a fixed Strouhal number calculation without using the exact Strouhal number will result in a non-zero unsteady residual.
Base Suction Coefficient - $C_{pb}$

193x81 grid

385x161 grid

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Airfoil - Motivation

- To demonstrate the ability of the NLFD method to time accurately resolve flows of engineering importance like a transonic pitching airfoil in inviscid and turbulent viscous environments.
  
  **Euler simulations**
  
  **Turbulent Navier-Stokes - Baldwin/Lomax model**

- To compare the efficiency of the NLFD solver to time accurate codes like UFLO82 which implements a dual time stepping technique.
## Airfoil - Experiment

<table>
<thead>
<tr>
<th>Description</th>
<th>Variable</th>
<th>Green</th>
<th>Davis</th>
<th>Landon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airfoil</td>
<td>0012</td>
<td>64A010</td>
<td>0012</td>
<td></td>
</tr>
<tr>
<td>Case Number</td>
<td>Slotted 119 134 Adaptive 201</td>
<td>CT Case 6 DI 55</td>
<td>CT Case 1</td>
<td></td>
</tr>
<tr>
<td>Mean Angle of Attack</td>
<td>$\alpha$</td>
<td>$-2.0^\circ - 6.0^\circ$</td>
<td>$0.0^\circ \pm 1.01^\circ$</td>
<td>$2.89^\circ \pm 2.41^\circ$</td>
</tr>
<tr>
<td>Reynolds Number</td>
<td>$Re_\infty$</td>
<td>$3.0 \times 10^6$</td>
<td>$12.56 \times 10^6$</td>
<td>$4.8 \times 10^6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$9.0 \times 10^6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mach Number</td>
<td>$M_\infty$</td>
<td>0.7</td>
<td>0.796</td>
<td>0.6</td>
</tr>
<tr>
<td>Reduced Frequency</td>
<td>$k_c$</td>
<td>0</td>
<td>0.202</td>
<td>0.0808</td>
</tr>
</tbody>
</table>
NACA 0012 Steady $C_l$

Euler

RANS

Coefficient of Lift

Angle of Attack (Degrees)

Run 119 – Pass 1
Run 119 – Pass 2
Run 134 – Pass 1
Run 134 – Pass 2
Run 201 – Pass 1
Run 201 – Pass 2
O EULER 161X33
O EULER 321X65
C EULER 257X33
C EULER 321X97

Coefficient of Lift

Angle of Attack (Degrees)

Run 119 – Pass 1
Run 119 – Pass 2
Run 134 – Pass 1
Run 134 – Pass 2
Run 201 – Pass 1
Run 201 – Pass 2
C VISCOUS 257X49
C VISCOUS 513X97
NACA 0012 Steady $C_m$

Euler

RANS

Coefficient of Moment

Angle of Attack (Degrees)

Run 119 – Pass 1
Run 119 – Pass 2
Run 134 – Pass 1
Run 134 – Pass 2
Run 201 – Pass 1
Run 201 – Pass 2
O Euler 161X33
O Euler 321X65
C Euler 257X33
C Euler 321X97
C Viscous 257X49
C Viscous 513X97
NACA 64A010 Unsteady $C_l$

Euler

RANS

Coefficient of Lift vs. Angle of Attack (Degrees)

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NACA 64A010 Unsteady $C_m$

Euler

RANS

Coefficient of Moment vs. Angle of Attack (Degrees)

AGARD–702: Davis, NLFD 1 HARMONIC, NLFD 2 HARMONIC, NLFD 3 HARMONIC
A study was carried out to compare the efficiency of UFLO82 and the NLFD solver.

Different time marching methods are commonly compared on the basis of the amount of error they produce. The comparison is constrained by advancing the solution forward an equivalent amount of time using an equal number of residual evaluations.

\[
\frac{du}{dt} = \lambda u
\]

\[
u_{n+1} = u_{n-1} + 2\Delta t \frac{du_n}{dt}
\]

\[
u_{n+1} = u_n + \frac{\Delta t}{2} \left( 3 \frac{du_n}{dt} - \frac{du_{n-1}}{dt} \right)
\]

This study compares the amount of work required by the different methods to obtain an equivalent level of error.
Residual Equivalence

- The difference in the solution ($\rho e$) between an NLFD and UFLO82 steady calculation is $\mathcal{O}(10^{-13})$. 
Error Definition

- The basis of comparison between the two codes will be the error in the magnitude of the Fourier coefficient of lift or moment.

\[
E_{N_l} = \left| |\hat{C}_{l_1}| - |\hat{C}_{l_{1512}}| \right|
\]

\[
E_{N_m} = \left| |\hat{C}_{m_1}| - |\hat{C}_{m_{1512}}| \right|
\]

- The most accurate solution was produced from a 512 points per period (SPP) UFLO82 calculation. The residual was driven to machine zero at each physical step which advanced the solution 48 complete periods. Only the final period was used for the purposes of quantifying the error.
We will use a more challenging test case for the numerical comparison. Reduced frequency is lower. Angle of attack variation is higher.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airfoil</td>
<td>NACA 64A010</td>
</tr>
<tr>
<td>Mean Angle of Attack</td>
<td>0.0°</td>
</tr>
<tr>
<td>Angle of Attack Variation</td>
<td>±2.0°</td>
</tr>
<tr>
<td>Mach Number</td>
<td>0.8</td>
</tr>
<tr>
<td>Reduced Frequency</td>
<td>0.05</td>
</tr>
</tbody>
</table>

All numerical computations will be carried out on a 161x33 O-mesh. The topology is constrained by the selection of the UFLO82 code.
Error between various temporal resolutions of the UFLO82 code and the control solution.
Asymptotic error in $C_{l_1}$ and $C_{m_1}$ computed at various temporal resolutions by the UFLO82 and NLFD codes.

**$C_l$ Error**

<table>
<thead>
<tr>
<th>Constant*Time Step</th>
<th>UFLO82</th>
<th>NLFD - 1 Harmonic</th>
<th>NLFD - 2 Harmonics</th>
<th>NLFD - 3 Harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-2}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-1}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-2}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-1}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**$C_m$ Error**

<table>
<thead>
<tr>
<th>Constant*Time Step</th>
<th>UFLO82</th>
<th>NLFD - 1 Harmonic</th>
<th>NLFD - 2 Harmonics</th>
<th>NLFD - 3 Harmonics</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-2}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10^{-1}</td>
<td></td>
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<tr>
<td>10^{-2}</td>
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</tr>
<tr>
<td>10^{-1}</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NLFD</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>UFLO82</td>
<td>45</td>
<td>125</td>
<td>244</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NLFD</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>UFLO82</td>
<td>18</td>
<td>45</td>
<td>123</td>
</tr>
</tbody>
</table>
Cost Estimates

\[ \frac{3W_{n+1} - 4W_n + W_{n-1}}{2\Delta t} + R(W_{n+1}) = 0 \]

\[ ikV\hat{W}_k + \hat{R}_k = 0 \]
Cost Comparison - $C_l$

- Cost comparison using the error in $C_l$ as the figure of merit.

<table>
<thead>
<tr>
<th>NLFD Modes</th>
<th>Cost</th>
<th>UFLO82 SPP</th>
<th>Cost</th>
<th>Cost Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3.06 \times 44 = 135$</td>
<td>45</td>
<td>$4 \times 45 \times 6 = 1080$</td>
<td>8.0</td>
</tr>
<tr>
<td>2</td>
<td>$5.13 \times 60 = 308$</td>
<td>125</td>
<td>$5 \times 125 \times 6 = 3750$</td>
<td>12.2</td>
</tr>
<tr>
<td>3</td>
<td>$7.30 \times 72 = 526$</td>
<td>244</td>
<td>$7 \times 244 \times 6 = 10248$</td>
<td>19.5</td>
</tr>
</tbody>
</table>

\[
\frac{\text{Time} \ n^{th}}{\text{Time} \ O^{th}} \times \text{MultigridCycles} \\
\text{Periods} \times \frac{\text{Solutions}}{\text{Period}} \times \frac{\text{MultigridCycles}}{\text{Solution}}
\]
Cost Comparison - \( C_m \)

Cost comparison using the error in \( C_m \) as the figure of merit.

<table>
<thead>
<tr>
<th>NLFD Modes</th>
<th>Cost ( \times ) SPP</th>
<th>UFLO82 Cost</th>
<th>Cost Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 3.06 \times 37 = 113 )</td>
<td>18</td>
<td>( 3 \times 18 \times 6 = 324 )</td>
</tr>
<tr>
<td>2</td>
<td>( 5.13 \times 50 = 257 )</td>
<td>45</td>
<td>( 4 \times 45 \times 6 = 1080 )</td>
</tr>
<tr>
<td>3</td>
<td>( 7.30 \times 65 = 475 )</td>
<td>123</td>
<td>( 6 \times 123 \times 6 = 4428 )</td>
</tr>
</tbody>
</table>

\[ \frac{\text{Time}_{n^{th}}}{\text{Time}_{O^{th}}} \times \text{MultigridCycles} \]

\[ \text{Periods} \times \frac{\text{Solutions}}{\text{Period}} \times \frac{\text{MultigridCycles}}{\text{Solution}} \]
Convergence Sensitivity

This research has also studied the sensitivity of the residual convergence of the NLFD solver on physical aspects of the problem. Plotted is the maximum of the absolute value of the residual over all the wavenumbers.

Dynamic Angle of Attack

Reduced Frequency
The Stanford ASCI project has already performed several unsteady viscous simulations of an experimental test rig using TFLO. Aachen turbine rig: Stator-rotor-stator : 6-7-6 model

The NLFD estimates are based on 14 temporal modes and 600 multigrid cycles to achieve convergence.

<table>
<thead>
<tr>
<th>Component</th>
<th>Multigrid Cycles</th>
<th>CPU Time (hrs)</th>
<th>Clock Time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TFLO</td>
<td>123,630</td>
<td>371,000</td>
<td>1985¹</td>
</tr>
<tr>
<td>NLFD</td>
<td>17,400</td>
<td>52,200</td>
<td>279¹</td>
</tr>
</tbody>
</table>

¹ Clock time based on 187 processors.
NLFD methods can be integrated into robust solvers exhibiting convergence performance close to that of more established steady state solvers.

Multigrid should be modified with coarse grid spectral viscosity to mitigate the dependence of convergence rates on unsteady effects.

The implicit treatment of the temporal derivative commonly found in dual time stepping codes is not recommended for NLFD solvers.

Residual averaging should be modified with CFL limiters to ensure stability on coarse grids within the multigrid cycle.
Conclusion - Cylinder

- Three time varying modes are required to predict the Strouhal number and base suction coefficient to engineering accuracy for the entire range of laminar Reynolds numbers.

- The GBVTP method can be used to predict the time period of the fundamental harmonic.

- The GBVTP method improved the agreement between the numerical and experimental results for base suction coefficient in comparison to fixed time period methods.
Conclusion - Pitching Airfoil

- Using just one time varying mode, the NLFD predictions for coefficient of lift provide excellent agreement with experimental results.

- In the case of the coefficient of moment, NLFD predictions provide poor agreement with experimental studies.

- Given equivalent spatial discretizations and overall error, the NLFD method is significantly more efficient than dual time stepping codes like UFLO82.
Contributions

- The first to demonstrate a frequency domain solver for the laminar and turbulent Navier-Stokes equations.
- Independently developed an NLFD solver with convergence rates equivalent to state-of-the-art steady codes.
- Enabling technologies: Residual Averaging, Spectral Viscosity
- Multiple topologies: Parallel multiblock, O-mesh and C-mesh grids
- Multiple equations: Euler, laminar and turbulent Navier-Stokes
- The first to demonstrate the Gradient Based Variable Time Period (GBVTP) method.
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