Aerospace Computing Laboratory

Stanford University

Airfoil Design Optimization Using Reduced Order Models Based on Proper Orthogonal Decomposition (POD)

Patrick A. LeGresley

Juan J. Alonso

Stanford University

Department of Aeronautics and Astronautics

Industrial Affiliates Meeting

April 26, 2000

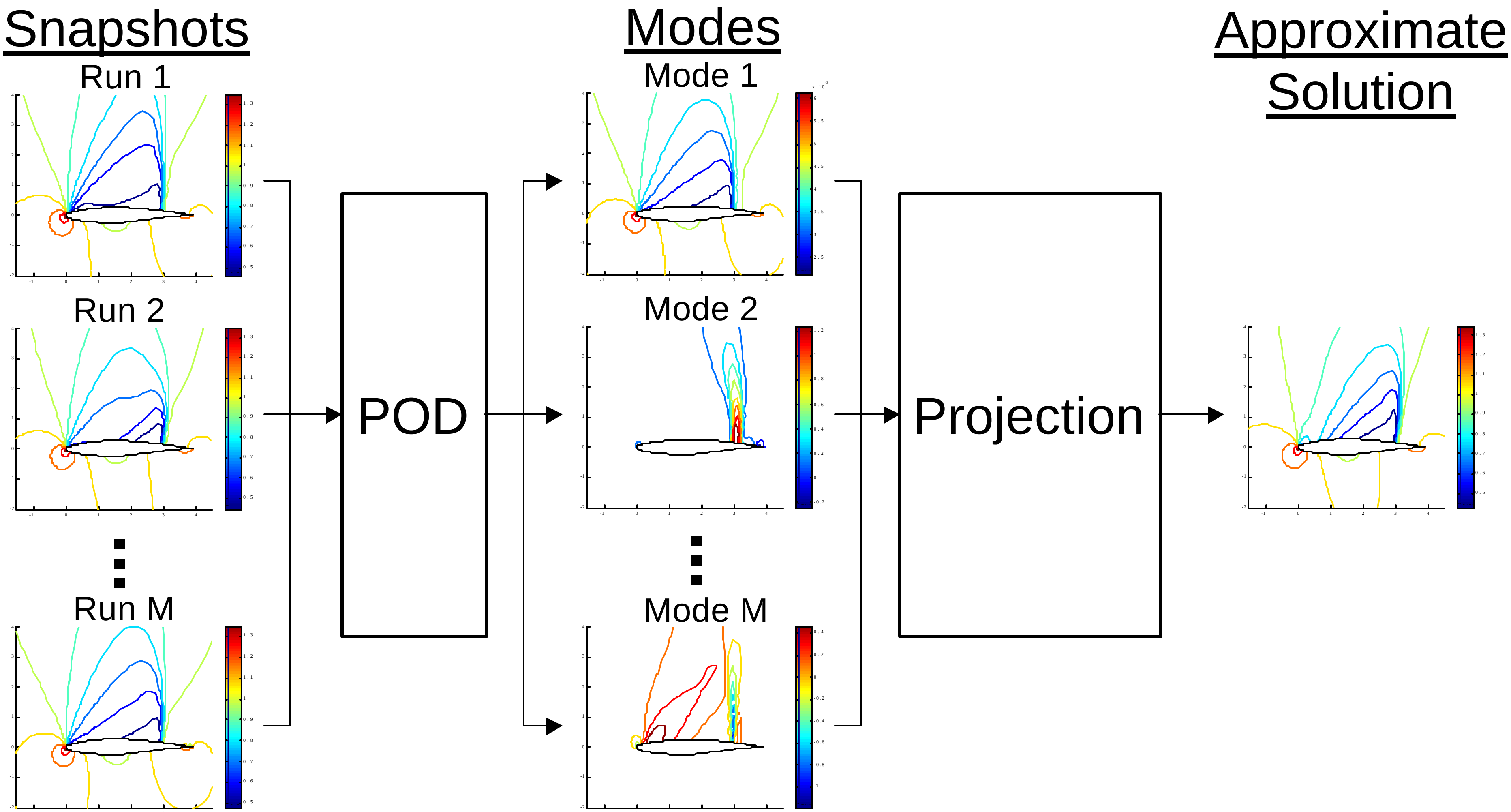
Motivation

- There is a need for high-fidelity models in the design & optimization of aerospace systems, but the computational cost is high
- To be computationally feasible the order of the models will have to be reduced (e.g. aeroelastic system with millions of degrees of freedom) while retaining “nice” properties - accuracy, equivalence in the limit, etc.
- POD is being investigated as a means to form approximate, reduced order models for use in the design environment

Motivation (cont.)

- Although our final goal is to use POD based models on MDO problems, we are initially applying it to aerodynamic shape optimization (ASO) problems – the parameterization of a surface and the necessary changes to obtain optimum behavior
- Within this framework, POD based models can be used both as a means to eliminate system analysis and a way to compute sensitivities
- In this work, we are exploring the use of these models to eliminate system analysis

Reduced Order Modeling



Proper Orthogonal Decomposition (POD)

- POD produces the *"best"* linear representation (basis modes) for a system of functions / flowfields (snapshots)
- For discrete systems, POD is an eigenvalue/vector problem
- In the case of airfoil design, the *snapshots* are *flow solutions* in which the design variables of interest (e.g. shape) are varied
- Flow solutions could come from function evaluations made during a typical optimization process. These additional function evaluations could then be used to progressively refine the POD model

POD Theory

- Starting with a collection of functions *u(x)*, we seek representations of the form:

$$u_N(x) = \sum_{j=1}^{N} a_j \varphi_j(x)$$

- Choosing φ to maximize the projection of *u* onto φ:

$$\max_{\varphi \in L^2} \frac{\left\langle \left|(u,\varphi)\right|^2 \right\rangle}{\|\varphi\|^2}$$

- Maximizing $\left\langle |u,\varphi|^2 \right\rangle$ subject to $\|\varphi\|^2 = 1$ yields:

$$\int \left\langle u(x) u^*(x') \right\rangle \varphi(x') dx' = \lambda \varphi(x) \qquad R = \left\langle u \otimes u^* \right\rangle$$

  - Eigenvectors of correlation matrix, *R*, are the principal axes
- Maximum corresponds to the largest eigenvalue:

$$\Re \varphi = \lambda \varphi$$

- Now we can represent the function by the first *M* modes:

$$u_N(x) = \sum_{j=1}^{N} a_j \varphi_j(x) \cong \sum_{j=1}^{M} a_j \varphi_j(x)$$

POD Theory – Method of Snapshots

- Reduces the *N x N* eigenvalue problem to an *M x M* problem (*N* can be a very large number)
- If φ is an eigenvector:

$$j = \sum_{k=1}^{M} a_k u^k$$

- Can find coefficients a_k such that

$$\sum_{k=1}^{M} \frac{1}{M}(u^i, u^k) a_k = l a_i$$

  – where *M* is the number of snapshots, *N* >> *M*

Projection

- Projection is the process of determining how to combine the modes such that both the governing equations and boundary conditions of the system are satisfied:

$$u_N(x) \cong \sum_{j=1}^{M} a_j j_j(x)$$

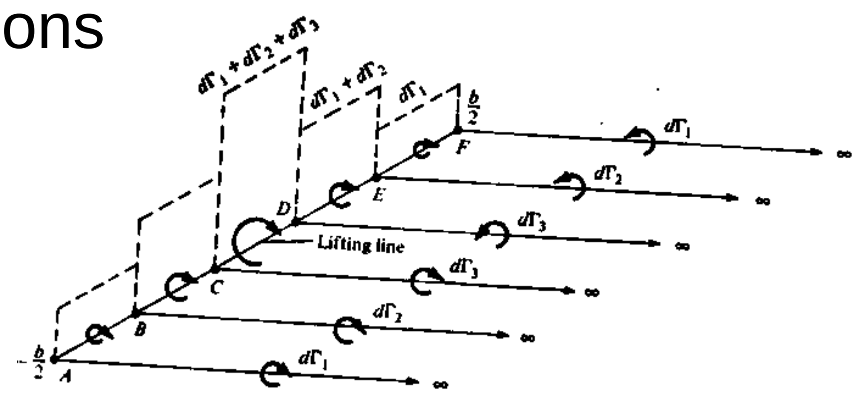
- Because the order of the model has been reduced, there are no longer enough degrees of freedom to exactly satisfy the governing equations and the boundary conditions
- Least squares method used to minimize the error of the approximate solution

Projection for 2-D Euler Flow

- Boundary conditions:
  - Flow tangency and pressure gradient at the surface
  - Free-stream conditions at the far field
- Governing equations:
  - Conservation of mass, momentum, and energy
- In this work we *only* consider the boundary conditions for computing the modal coefficients
- The boundary conditions result in a non-linear system of equations for the coefficients of the modal expansion
- The equations can be linearized by separating the flow tangency and pressure gradient boundary conditions

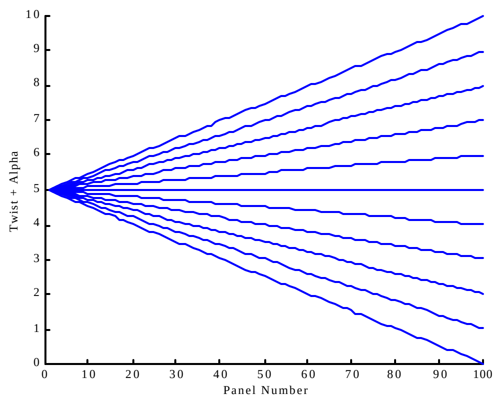
Panel Method

- Simple panel method initially considered as a system for getting familiar with POD
- Panel method is a means of calculating the lift distribution of a wing, which is a function of the twist distribution
- Snapshots in which the twist distribution was varied were used to compute modes
- The modes were used to estimate the lift distribution for new twist distributions



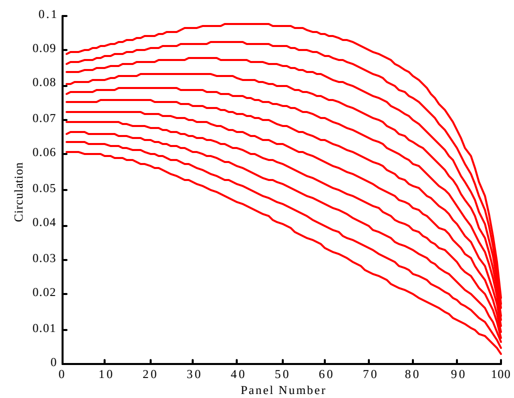
Panel Method Results

11 Twist Distributions



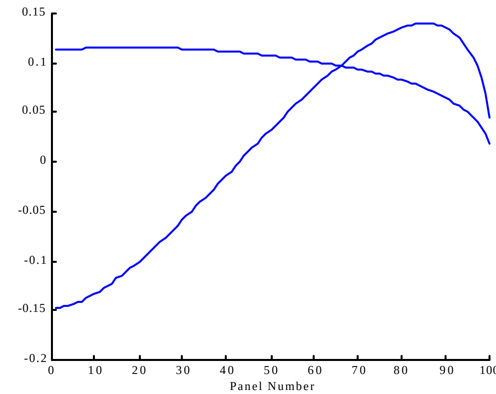
Panel Code

Circulation Distributions

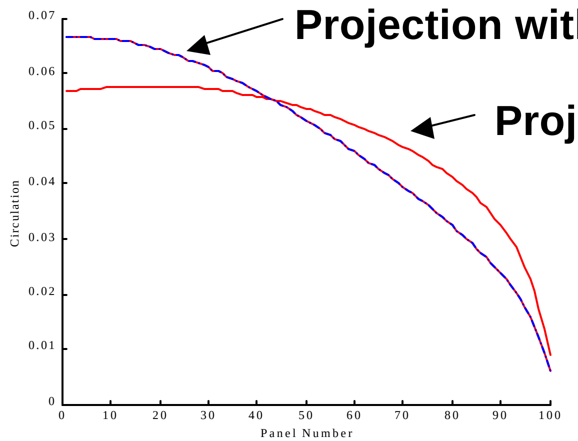


POD

Modes



Projection

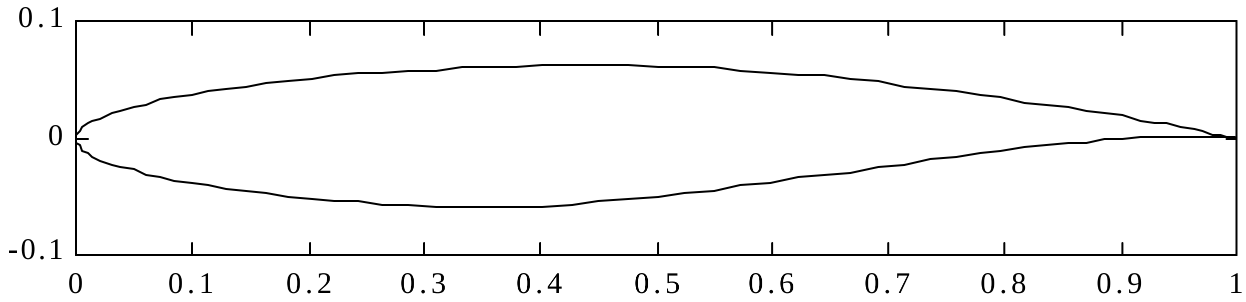


Why does the use of only two modes give an exact solution ?

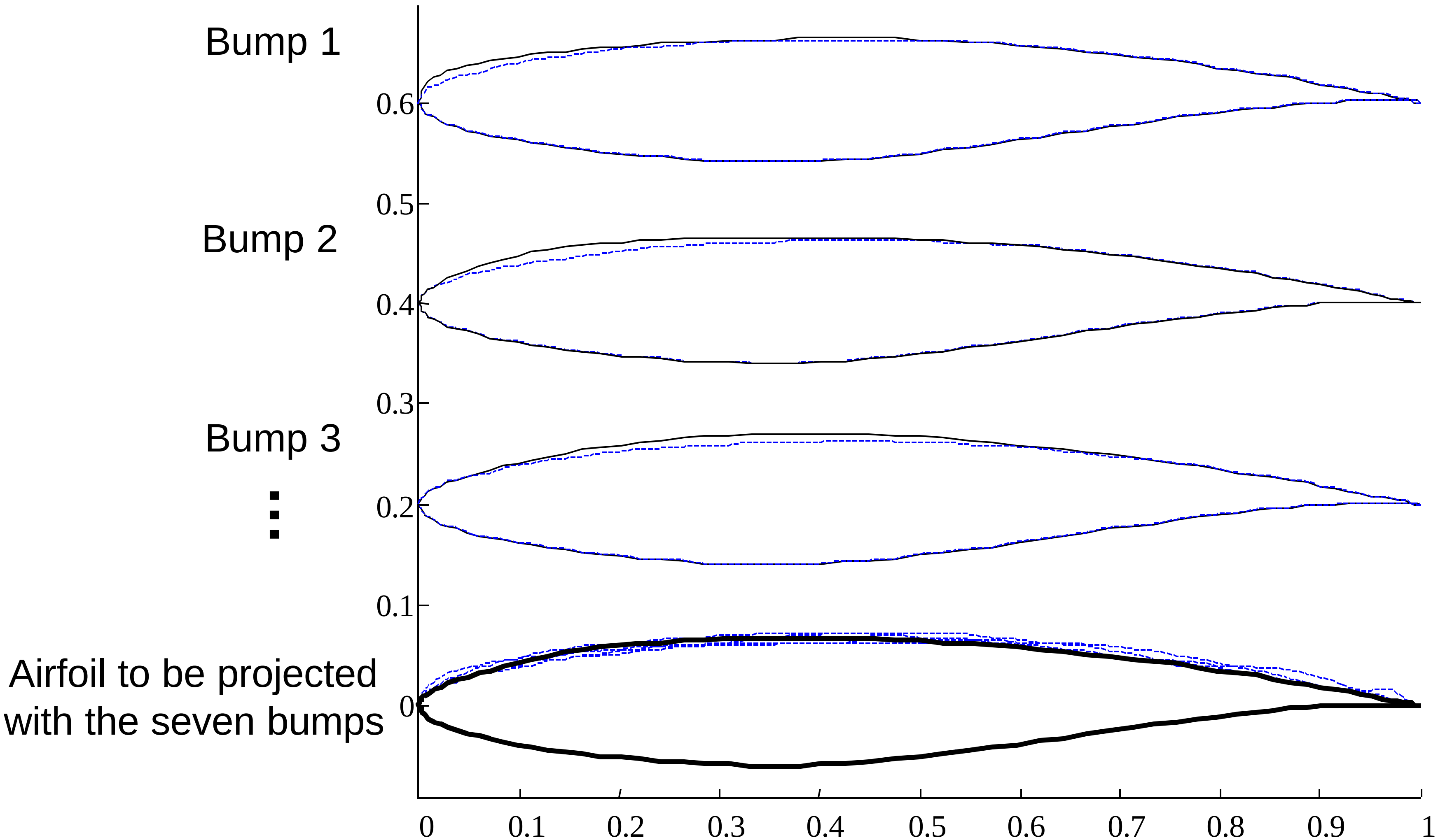
-Basic and Additional Lift Distribution

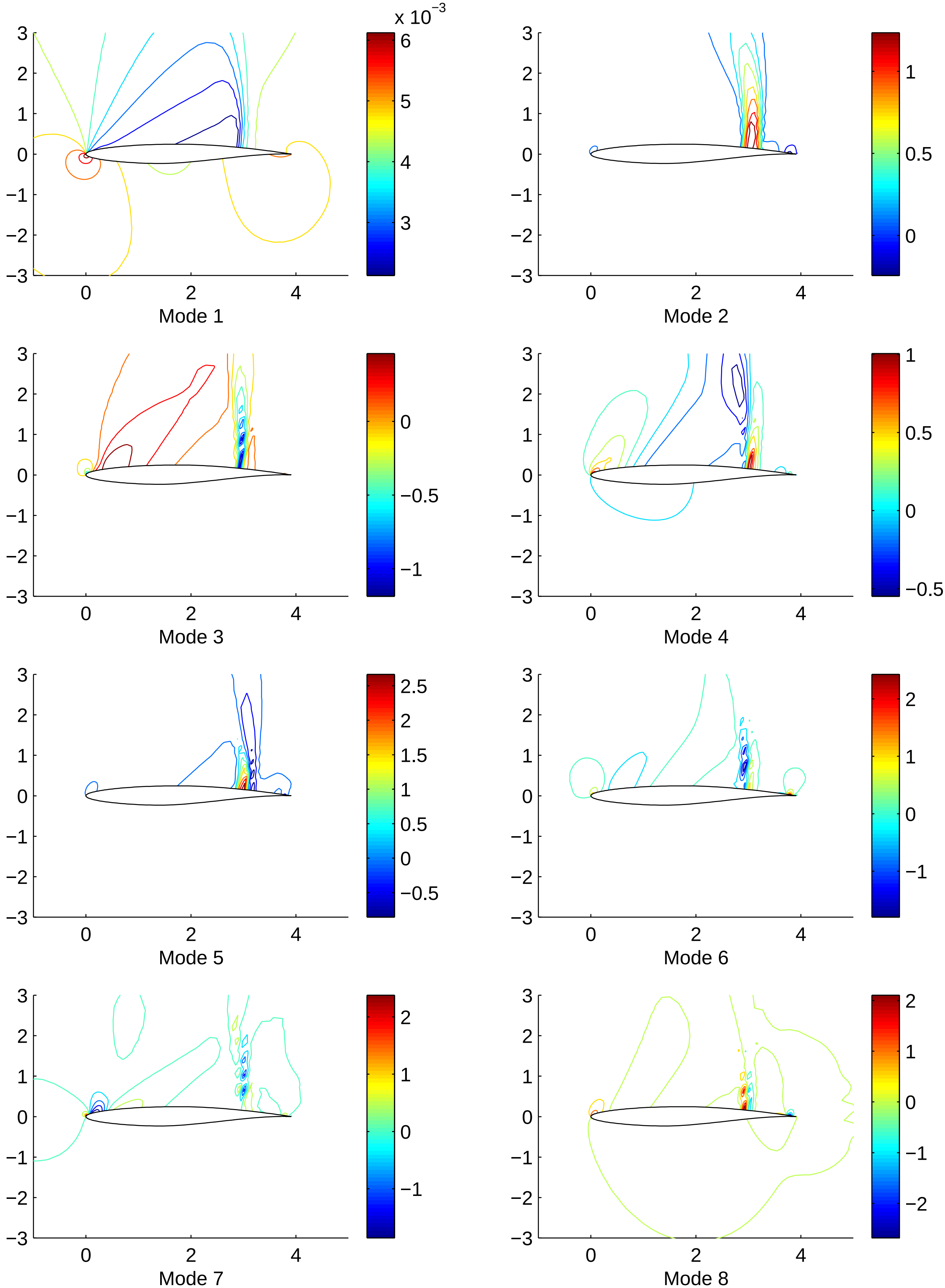
Transonic 2-D Euler Flow

- Projecting changes in geometry of the upper surface of the RAE 2822 airfoil at a free-stream Mach number of 0.75
- Snapshots consist of eight flow solutions:
  - One with the baseline airfoil
  - Seven in which a series of bumps are distributed over the upper surface of the airfoil
- Projecting a flow solution for an airfoil with a single bump with a height of 80% of those used in the snapshots



Transonic 2-D Euler Flow (cont.)

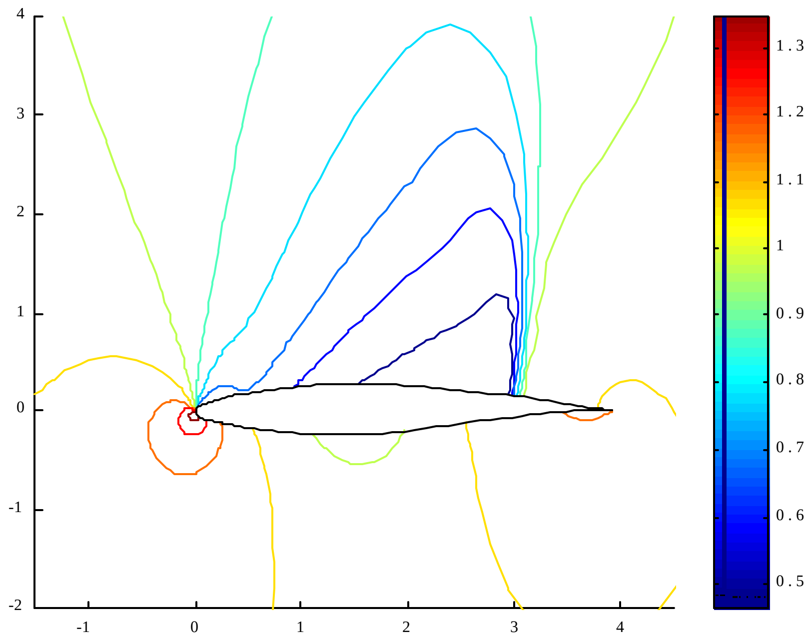




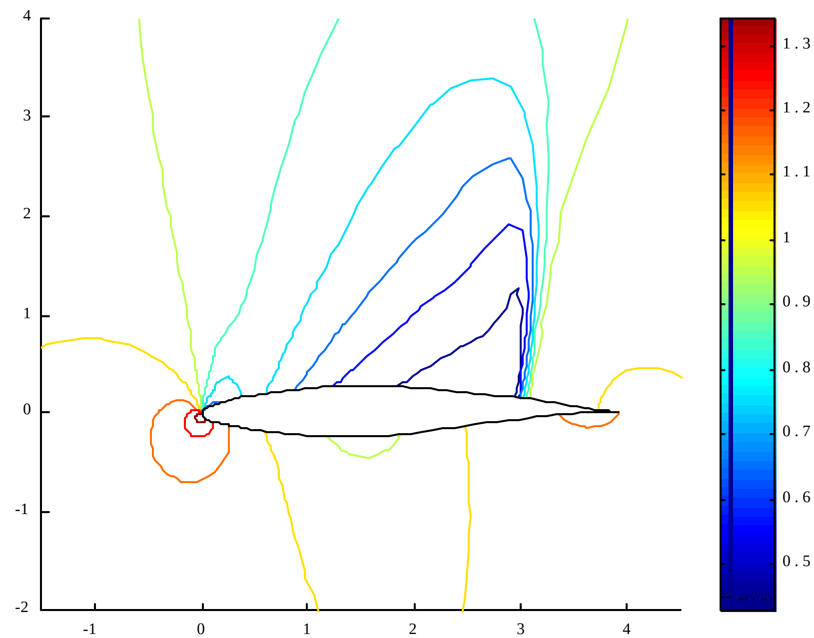
x 10^-3
Mode 1
Mode 2
Mode 3
Mode 4
Mode 5
Mode 6
Mode 7
Mode 8

Transonic 2-D Euler Flow (cont.)

- Computational time reduced from 9.8 sec to 0.8 sec
- Pressure contours can be used to see how well the flow solution in the domain is estimated



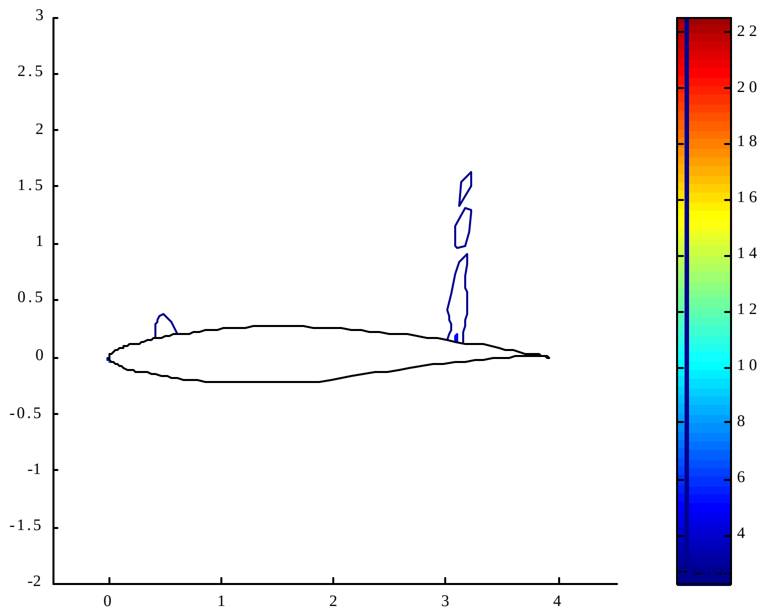
Exact



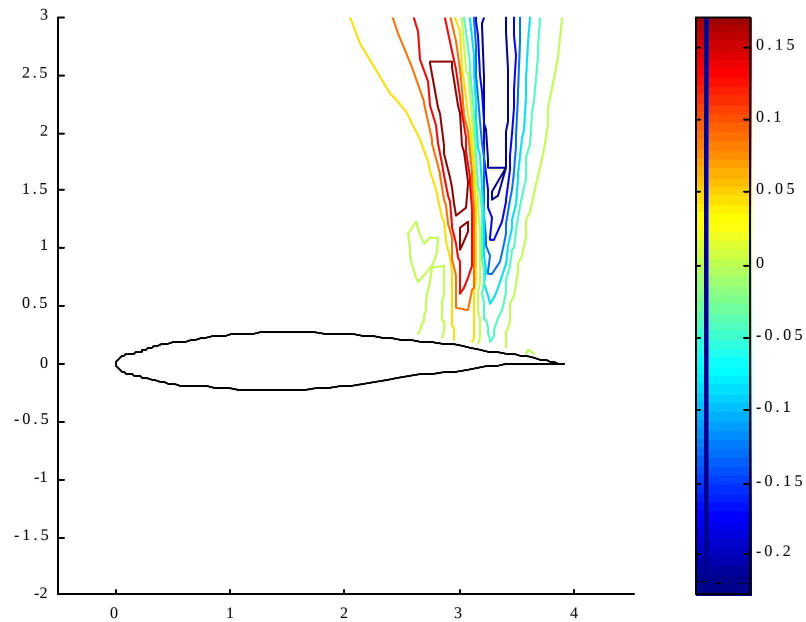
Projected

Transonic 2-D Euler Flow (cont.)

- Error contours, and residuals – a measure of how closely the solution satisfies the governing equations



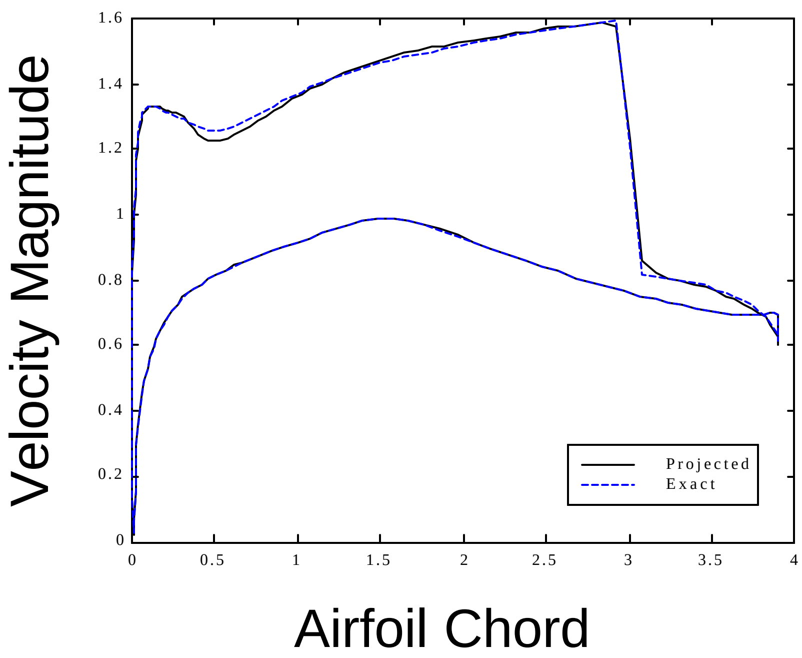
Percentage Error
in Velocity Magnitude

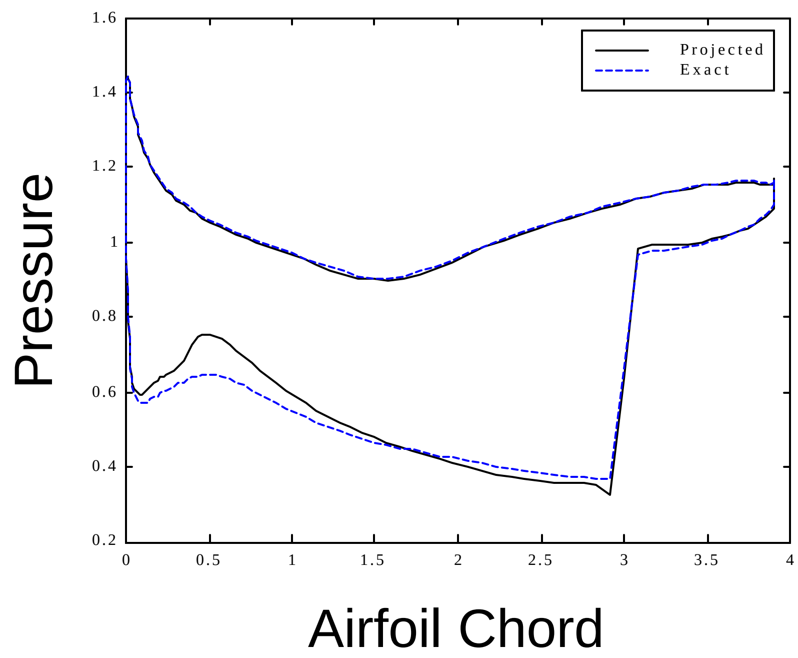


Mass Residual

Transonic 2-D Euler Flow (cont.)

- The accuracy of gradients will depend on the flow solution at the surface





Conclusions & Future Work

- Initial results are very encouraging (including other test cases at different Mach numbers)
- Reduced order models based on POD could be a computationally efficient means of approximating a system
- Accuracy of gradients needs to be assessed for use in design optimization
- Ultimately we desire to perform design optimization on coupled disciplines (e.g. aeroelastics) using these reduced order models
- Much work remains to be done