A GRADIENT ACCURACY STUDY FOR THE ADJOINT-BASED NAVIER-STOKES DESIGN METHOD

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37th AIAA Aerospace Sciences Meeting
Reno, NV
January, 1999

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Historical perspective

- **Motivation** for **Aerodynamic Shape Design**
  - Better Performance (ex. Higher L/D, Less Drag).
  - Less Cost, Less Time, Less Labor.
  - More Physical, More Accurate.
- **Experiment**
  - Has been used as a main design tool since 1920's (ex. Wind Tunnel Test).
  - Cost, Time and Labor intensive.
- **Theoretical aerodynamics**
  - Has matured in parallel with experiments (ex. Thin Airfoil Theory by Ludwig Prandtl).
  - Less Practical for higher level of flow models. (ex Compressibility, Viscosity, and Turbulence).
- **CFD** as an **Analysis Tool** in **Design Process** since 1960's
  - Has reduced the Time and Cost.
  - Has included more Physics with better Accuracy.

- **CFD** as a **Design Tool**
  - Has been coupled directly to **Numerical Optimization processes** (ex. **Drag Minimization**).
  - **Gradient Based Optimization**
    - **Finite Difference Method**
      - Transonic aerodynamic shape design using the potential flow equations by Hicks, Murman, and Vanderplaats in 1974.
    - **Control Theory Approach** or **Adjoint Method**
      - For shape design using elliptic equations by Pironneau in 1984.
    - **Continuous Adjoint Method**
      - **Adjoint Formulation First** and **Discretization Next**
      - First introduced to transonic airfoil and wing design using the transonic **potential** flow equation by Jameson in 1988.
      - Using the **Euler** equations by Jameson and Reuther in 1994.
      - Complete configurations design using the **Navier-Stokes** equations for the flow model with an **Inviscid Version of the Adjoint** for Gradients by Jameson, Pierce, and Martinelli in 1997.
    - **Discrete Adjoint Method**
      - **Discretization First** and **Adjoint Formulation Next**
      - Using Navier-Stokes equations by Anderson and Venkatakrishnan in 1998.
    - **ADIFOR** : Automatic Differentiation of Fortran (Hu in 1997)

- **Non Gradient Based Method**
  - **GA** : Genetic Algorithm method (Crispin in 1994)
  - **DISC** : Direct Iterative Surface Curvature (Campbell)

Objective

- Implementation of a **Continuous Adjoint Design Method** that uses the **Navier-Stokes** equations as a flow model.
- Verification of the **Accuracy** and **Efficiency** of the present **Continuous Adjoint Method** by the comparison with **Gradients** from **Finite Difference Method**.
- Demonstration with **Preliminary Examples**.

Summary of Aerodynamic Shape Optimization

In gradient-based optimization technique,

- **Cost** (or **Objective**) function (**Drag, L/D** etc. for example) is set to be **Minimized** (or **Maximized**).
- **Control** function(**Airfoil Shape** for example) is parameterized by a set of **Design Variables** (**Mesh Points** , **Hick-Henne's Sine Bump** function for example)
- **Constraint** (the flow equations for aerodynamic design) is introduced to express the **Dependence** of the **Cost** function and the **Control** function.
- **Sensitivity Derivatives** (or **Gradients**) are calculated. **Gradients** refer to **Changes** in the **Cost** function with respect to **Changes** in the **Design Variables**. Large computational savings are derived from use of **Adjoint Method**.
- Shape is improved by **Search Procedure** with a suitable **Optimization** Algorithm.

Symbolic Description of Continuous Adjoint Method

Let I be the **cost** (or **objective**) function

$$I = I(w, \mathcal{F})$$

where

$$\begin{aligned} w &= \text{flow field variables} \\ \mathcal{F} &= \text{design variables} \end{aligned}$$

The **first variation** of the cost function is

$$\delta I = \frac{\partial I}{\partial w}^T \delta w + \frac{\partial I}{\partial \mathcal{F}}^T \delta \mathcal{F}$$

The **flow field equation** and its **first variation** are

$$R(w, \mathcal{F}) = 0$$

$$\delta R = 0 = \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F}$$

Introducing a **Lagrange Multiplier**, ψ, and using the **flow field equation** as a **constraint**

$$\begin{aligned}\delta I &= \frac{\partial I}{\partial w}^T \delta w + \frac{\partial I}{\partial \mathcal{F}}^T \delta \mathcal{F} - \psi^T \left\{ \left[\frac{\partial R}{\partial w}\right] \delta w + \left[\frac{\partial R}{\partial \mathcal{F}}\right] \delta \mathcal{F} \right\} \\ &= \left\{ \frac{\partial I}{\partial w}^T - \psi^T \left[\frac{\partial R}{\partial w}\right] \right\} \delta w + \left\{ \frac{\partial I}{\partial \mathcal{F}}^T - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right\} \delta \mathcal{F}\end{aligned}$$

By choosing ψ such that it satisfies the **adjoint equation**

$$\left[\frac{\partial R}{\partial w}\right]^T \psi = \frac{\partial I}{\partial w},$$

we have

$$\begin{aligned}\delta I &= \left\{ \frac{\partial I}{\partial \mathcal{F}}^T - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}}\right] \right\} \delta \mathcal{F} \\ &= \mathcal{G}^T \delta \mathcal{F}\end{aligned}$$

This reduces the **gradient**($\mathcal{G}$) calculation for an **arbitrarily large number of design variables** at a **single design point** to

One Flow Solution

$+$ **One Adjoint Solution**

Adjoint vs. Finite Difference Methods

- Dependence on δw
  - **Finite Difference** Method : **Dependent** $\Rightarrow N_d + 1$ Flow Calculations
  - **Continuous Adjoint Method** : **Independent** $\Rightarrow$ 1 Flow + 1 Adjoint Calculations
- Convergence Tolerance Issue of **Finite Difference** Method

$$\begin{aligned} G &= \frac{dI}{d\mathcal{F}} \\ &= \frac{(I + \triangle I \pm \mathcal{E}) - (I \pm \mathcal{E})}{\triangle \mathcal{F}} \\ &= \frac{\triangle I}{\triangle \mathcal{F}} \left(1 \pm \frac{\mathcal{E}}{\triangle I}\right), \end{aligned}$$

where $\mathcal{E}$ is error in numerical computation.
Let's say $\mathcal{E} \sim R_a$ and $\triangle I = \triangle C_d$, and for a good approximation, at least,

$$\frac{\mathcal{E}}{\triangle I} < 10^{-2},$$

then for one count drag($\triangle C_d = .0001$),

$$R_a < 10^{-6}.$$

- Step Size Issue of **Finite Difference** Method
  For a good Finite Difference approximation, the Step Size $\triangle\mathcal{F}$ should be small like
  $$\triangle\mathcal{F} < 10^{-4}.$$
  Now let's say again
  $$\frac{\mathcal{E}}{\triangle I} < 10^{-2}$$
  and suppose we have
  $$\mathcal{E} \sim R_a \sim 10^{-6}$$
  and
  $$G \sim 1,$$
  then
  $$\triangle\mathcal{F} \sim \triangle I > 10^{-4}.$$
  $\Rightarrow$ Contradiction!.
- Moreover, since I is an integral function along the boundary,
  $$\mathcal{E} \sim R_b,$$
  and since $R_b \gg R_a$, especially in **viscous** calculations,
  the Convergence and Step Size problems are **more severe**.

- Convergence and Step Size Issue of **Continuous Adjoint Method**

  - Since the **Continuous Adjoint Gradient** equation,

$$\begin{aligned} \delta I &= \left\{ \frac{\partial I}{\partial \mathcal{F}}^T - \psi^T \left[\frac{\partial R}{\partial \mathcal{F}} \right] \right\} \delta \mathcal{F} \\ &= \mathcal{G}^T \delta \mathcal{F}, \end{aligned}$$

    is **analytically** formulated, **Not finite differenced**,
    Gradient of **Continuous Adjoint Method** is **Independent** of **Step Size**.
    ⇒ **Robustness, Reliability** and **Consistency**.

  - Since $\mathcal{G}$ is f(w,ψ),

$$\mathcal{E}_{adjoint} \sim \max(R_{flow}, R_{costate}).$$

    ⇒ **Less Iterations** for each Flow or Adjoint Calculation.
    ⇒ **Efficiency** of **Continuous Adjoint Method**.

Numerical Optimization Method

The **search procedure** used in this work is a simple **descent method** in which small steps are taken in the negative gradient direction.

$$\delta \mathcal{F} = -\lambda \mathcal{G},$$

where λ is positive and small enough that the first variation is an accurate estimate of δI. Then

$$\delta I = -\lambda \mathcal{G}^T \mathcal{G} < 0.$$

After making such a modification, the gradient can be recalculated and the process repeated to follow a path of **steepest descent** until a minimum is reached.

Implementation of Navier–Stokes Design

- The design algorithm has four distinct modules:
  1. **Flow Solver**
  2. **Adjoint Solver**
  3. **Geometry Modification** and **Mesh Perturbation** Algorithm
  4. **Optimization** Algorithm
- The design procedures can be summarized as follows:
  1. Solve the flow equations for ρ, u_1, u_2,u_3, p.
  2. Solve the adjoint equations for ψ subject to appropriate boundary conditions.
  3. Evaluate $\mathcal{G}$.
  4. Update the shape based on the direction of steepest descent.
  5. Return to 1.

Navier–Stokes Equation

$$\frac{\partial w}{\partial t} + \frac{\partial f_i}{\partial x_i} = \frac{\partial f_{vi}}{\partial x_i} \quad \text{in } \mathcal{D}, \tag{1}$$

$$w = \begin{Bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{Bmatrix}, f_i = \begin{Bmatrix} \rho u_i \\ \rho u_i u_1 + p\delta_{i1} \\ \rho u_i u_2 + p\delta_{i2} \\ \rho u_i u_3 + p\delta_{i3} \\ \rho u_i H \end{Bmatrix}, \quad f_{vi} = \begin{Bmatrix} 0 \\ \sigma_{ij}\delta_{j1} \\ \sigma_{ij}\delta_{j2} \\ \sigma_{ij}\delta_{j3} \\ u_j\sigma_{ij} + k\frac{\partial T}{\partial x_i} \end{Bmatrix}. \tag{2}$$

$$p = (\gamma - 1)\,\rho \left\{ E - \frac{1}{2}(u_i u_i) \right\},$$

The viscous stresses:

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k},$$

$$k = \frac{\gamma \mu}{Pr}, \quad T = \frac{p}{(\gamma - 1)\rho}.$$

By using the **Coordinate Transformation** Matrix

$$K_{ij} = \left[\frac{\partial x_i}{\partial \xi_j}\right], \quad J = \det(K), \quad K_{ij}^{-1} = \left[\frac{\partial \xi_i}{\partial x_j}\right],$$

the Navier-Stokes equations can then be written in computational space as

$$\frac{\partial (Jw)}{\partial t} + \frac{\partial (F_i - F_{vi})}{\partial \xi_i} = 0 \text{ in } \mathcal{D}, \tag{3}$$

where $F_i = S_{ij} f_j$ and $F_{vi} = S_{ij} f_{vj}$, $S_{ij} = JK_{ij}^{-1}$.

We have made use of the property that

$$\frac{\partial S_{ij}}{\partial \xi_i} = 0 \tag{4}$$

FLO103 Navier-Stokes Solver

- **Methodology**
  - **Runge-Kutta Explicit Time Stepping**
  - **Cell Centered Spatial Discretization**
  - **Jameson-Schmidt-Turkel(JST) Scheme** with **Adaptive Coefficients** for **Artificial Dissipation**
  - **Local Time Stepping**
  - **Implicit Residual Smoothing**
  - **Multigridding**
- Turbulence
  - **Baldwin-Lomax** Model
  - **Freezing** Eddy Viscosity

Adjoint Equation

Suppose that the performance is measured by a **Cost Function**

$$I = \int_{\mathcal{B}} \mathcal{M}\left(w, \mathcal{F}\right) dB_{\xi} + \int_{\mathcal{D}} \mathcal{P}\left(w, \mathcal{F}\right) dD_{\xi},$$

where $\mathcal{F}$ represents the **Design Variables**.
A shape change produces a variation

$$\delta I = \int_{\mathcal{B}} \delta\mathcal{M}(w, \mathcal{F})\, dB_{\xi} + \int_{\mathcal{D}} \delta\mathcal{P}(w, \mathcal{F})\, dD_{\xi}. \tag{5}$$

Here δM and δP can be split into contributions associated with δw and $\delta\mathcal{F}$ using the notation

$$\begin{aligned} \delta\mathcal{M} &= \left[\mathcal{M}_w\right]_I \delta w + \delta\mathcal{M}_{II}, \\ \delta\mathcal{P} &= \left[\mathcal{P}_w\right]_I \delta w + \delta\mathcal{P}_{II}. \end{aligned} \tag{6}$$

In the steady state,

$$\frac{\partial}{\partial \xi_i} \delta \left(F_i - F_{vi}\right) = 0. \tag{7}$$

Here δF_i and δF_{vi} can also be split into contributions associated with δw and $\delta \mathcal{F}$ using the notation

$$\begin{aligned} \delta F_i &= \left[F_{iw}\right]_I \delta w + \delta F_{iII} \\ \delta F_{vi} &= \left[F_{viw}\right]_I \delta w + \delta F_{viII}. \end{aligned} \tag{8}$$

Multiplying by a co-state vector ψ

$$\int_{\mathcal{D}} \psi^T \frac{\partial}{\partial \xi_i} \delta \left(F_i - F_{vi}\right) d\mathcal{D}_\xi = 0. \tag{9}$$

If ψ is differentiable

$$\int_{\mathcal{B}} n_i \psi^T \delta \left(F_i - F_{vi}\right) d\mathcal{B}_\xi \tag{10}$$

$$- \int_{\mathcal{D}} \frac{\partial \psi^T}{\partial \xi_i} \delta \left(F_i - F_{vi}\right) d\mathcal{D}_\xi = 0. \tag{11}$$

and, subtracting from δI

$$\begin{aligned} \delta I &= \int_{\mathcal{B}} \left[\delta \mathcal{M} - n_i \psi^T \delta \left(F_i - F_{vi}\right)\right] d\mathcal{B}_\xi \\ &+ \int_{\mathcal{D}} \left[\delta \mathcal{P} + \frac{\partial \psi^T}{\partial \xi_i} \delta \left(F_i - F_{vi}\right)\right] d\mathcal{D}_\xi. \end{aligned} \tag{12}$$

ψ is an arbitrary differentiable function computed from the **Adjoint Equation**:

$$\frac{\partial \psi^T}{\partial \xi_i} \left[F_{iw} - F_{viw} \right]_I \; + \; \left[\mathcal{P}_w \right]_I = 0 \quad \text{in } \mathcal{D}. \tag{13}$$

with the **Adjoint Boundary Condition** :

$$n_i \psi^T \left[F_{iw} - F_{viw} \right]_I = \left[\mathcal{M}_w \right]_I \quad \text{on } \mathcal{B}. \tag{14}$$

The remaining terms from equation yield a simplified expression for the variation δI which defines the **Gradient**

$$\begin{aligned} \delta I \;&=\; \int_{\mathcal{B}} \left\{ \delta \mathcal{M}_{II} - n_i \psi^T \left[\delta F_i - \delta F_{vi} \right]_{II} \right\} d\mathcal{B}_\xi \\ &+\; \int_{\mathcal{D}} \left\{ \delta \mathcal{P}_{II} + \left[\delta F_i - \delta F_{vi} \right]_{II} \right\} d\mathcal{D}_\xi. \end{aligned} \tag{15}$$

The **Adjoint Equation** is **Linear** and **Steady**.
A **Time-Like** derivative is added and the adjoint solution is obtained by the **Same Methodology** as for flow solver.

Geometry Modification and Mesh Perturbation

- **Design Variable**
  - **Hicks-Henne's Sine Bump Function** :

$$b(x) = A\left[\sin\left(\pi x^{\frac{\log 5}{\log t_1}}\right)\right]^{t_2}, \qquad 0 \le x \le 1$$

    Here, A is the maximum bump magnitude, t_1 locates the maximum of the bump at $x = t_1$, and t_2 controls the width of the bump.
  - **Mesh Points**
  - **B-Splines**

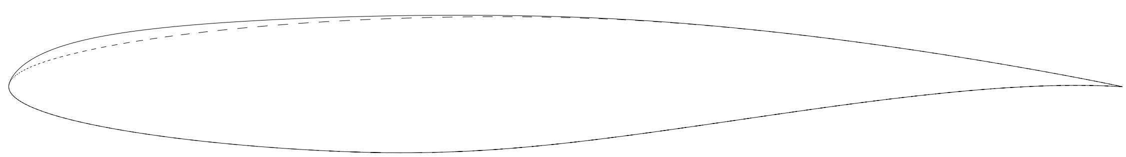


Figure 1: **Typical Sine Bump**

- **Mesh Perturbation**

  New grids are generated by shifting the grid points along the radial coordinate lines. The modification to the grid has the form

$$x^{new} = x^{old} + \mathcal{N}\left(x_s^{new} - x_s^{old}\right)$$

$$y^{new} = y^{old} + \mathcal{N}\left(y_s^{new} - y_s^{old}\right).$$

  Here,

$$\mathcal{N} = \frac{S_{total} - S_j}{S_{total}}.$$

Results

- **Euler Inverse Design Problem**
- **Navier-Stokes Inverse Design Problem**
- **Navier-Stokes Drag Minimization Problem**
- For Each Problem
  - **Gradient Study**
    1. **Finite Gradient** Study
       * **step size** issue
       * **flow convergence** issue
       * **mesh resolution** issue
    2. **Continuous Adjoint Gradient** Study
       * **step size** issue
       * **flow convergence** issue
       * **adjoint convergence** issue
       * **mesh resolution** issue
    3. **Gradient Comparison** Study
  - **Design Example**
- **Parallel Implementation Results**

Euler Inverse Design Problem

- **Cost function**
  - Pressure Difference from Desired Target Pressure at **Fixed** α,
$$I = \frac{1}{2} \int_{\mathcal{B}} (p - p_d)^2 \, dS.$$
  - Initial Airfoil : **NACA64A410**
  - Target Pressure : Pressure of **Korn** at **M=0.75, alpha=0.124**
- **Design Variables**
  - **54** of **Bump Functions**
- **Mesh**
  - **192 x 32** Mesh for Euler Calculation
- **Design Examples**
  - From **NACA64A410** to **Korn**
  - From **Korn** to **NACA64A410** at **M=0.75, alpha=0.**

Navier-Stokes Inverse Design Problem

- **Cost function**
  - Pressure Difference from Desired Target Pressure at **Fixed** α,
  $$I = \frac{1}{2} \int_{\mathcal{B}} (p - p_d)^2 \, dS.$$
  - Initial Airfoil : **Korn**
  - Target Pressure : Pressure of **NACA64A410** at **M=0.75, alpha=0.**
- **Design Variables**
  - **50** of **Bump Functions** for **Gradient** study
- **Mesh**
  - **512 x 64** Mesh for Navier-Stokes Calculation
- **Design Results**
  - From **Korn** to **NACA64A410** with **54 Bump Function** :
    $P_{error} = 0.0056$ in 100 Design Cycles
  - From **RAE2822** to **NACA64A410** with **54 Bump Function** :
    $P_{error} = 0.0043$ in 100 Design Cycles

Navier-Stokes Drag Minimization Problem

- **Cost function**
  - **Total Drag**($D_{viscous} + D_{pressure}$) at **Fixed CL**
  - Initial Airfoil : **RAE2822** at **CL = .84**
- **Design Variable**
  - **50** of **Bump Functions** for **Gradient** study
- **Mesh**
  - **512 x 64** Mesh for Navier-Stokes Calculation
- **Design Results**
  - **RAE2822 Total Drag Minimization** : -43% D_{total} Reduction in 17 Design Cycle
  - **RAE2822 Pressure Drag Minimization** : -63% $D_{Pressure}$ Reduction in 14 Design Cycle

Parallel Implementation

- **SPMD (Single Program Multiple Data)** Strategy.
- **MPI (Message Passing Interface)** Library for Message Passing.
- **MPI** is Used to Update the **Double Halo** Quantities at Every Stage of the Time-Stepping for Flow and Adjoint Solvers.

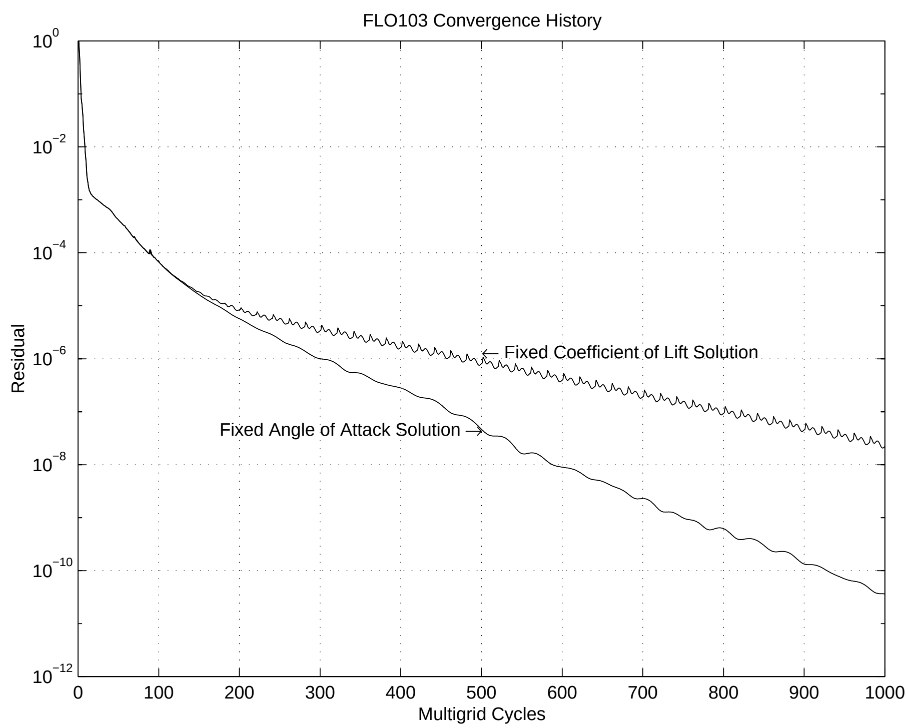
Conclusions

- **Navier-Stokes** Based Adjoints are implemented to Complement the **Viscous Design** Capability.
- The **Efficiency** and **Accuracy** of the present **Continuous Adjoint Method** are verified by comparison with **Gradients** from **Finite Difference Method**.
- The **Gradients** calculation results agree well with the **Numerical Error Analysis**.
- Adjoint Methods Demonstrate a **Very Large Gain in Computational Efficiency** Over Traditional Finite Difference Methods.
- Preliminary Design Examples for 2D Airfoil for **Inverse**, **Pressure Drag minimization** , and **Total Drag Minimization** has been Demonstrated with Viscous Flows.
- A **Further Reduction** in Wall Clock Time is Realized via **Parallel Computing** with **Near Linear Speed Up**.

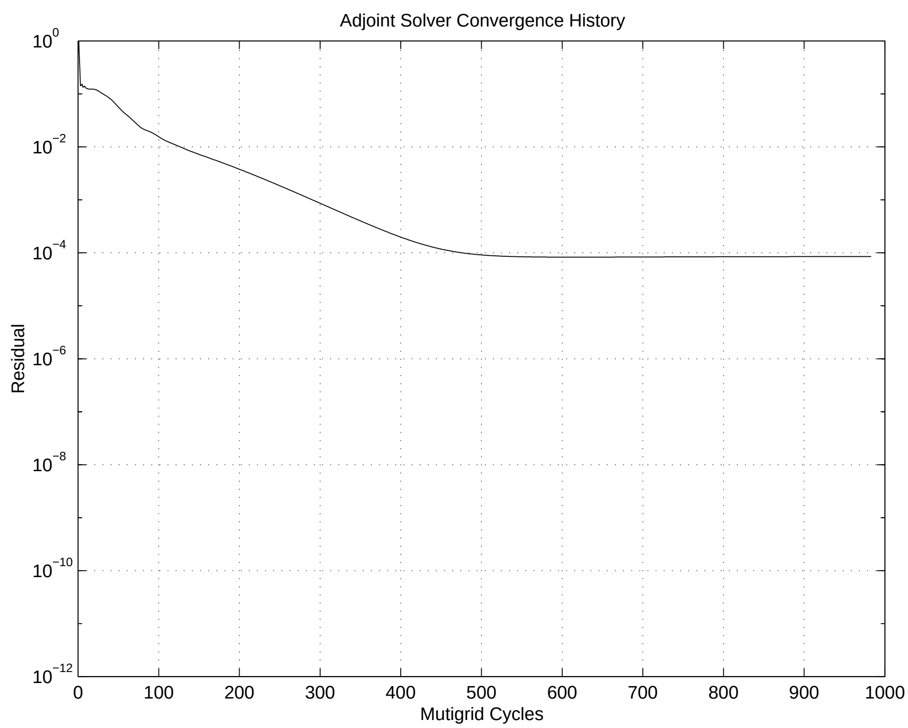
Future Work

- **Viscous Dominated** Problem Design(ex. Subsonic Airfoil, High Lift System)
- **3D** Application.
- Gradient Comparison study with other Methods (ex.**ADIFOR**,**Direct Method**)
- Further Study of **Adjoint Solver** and **Adjoint Boundary Conditions**.
- Search for other suitable **Optimization Algorithms** and Implementation.
- **Mesh Refinement** of Design Implementation(ex. Progressive Adjoint Design Method)
- Develop **Graphical User Interfaces** for Engineers to allow:
  - real time **visualization** of progressing design
  - **active control** for the design process
  - **understanding** of the vast datasets
  - a means of making **Educated Design Decisions**

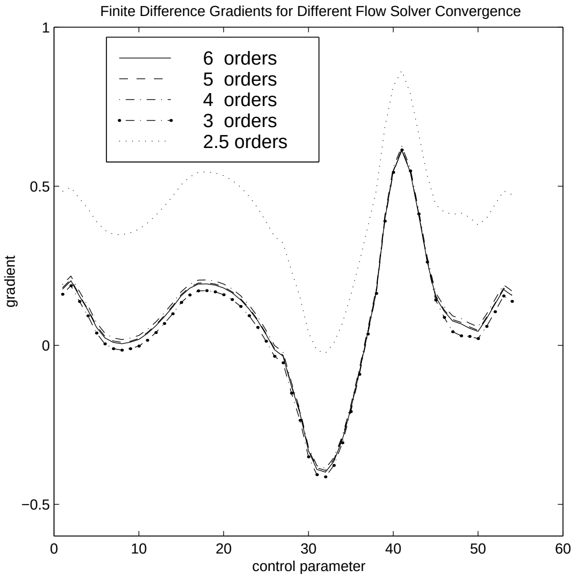
FLO103 CONVERGENCE HISTORY



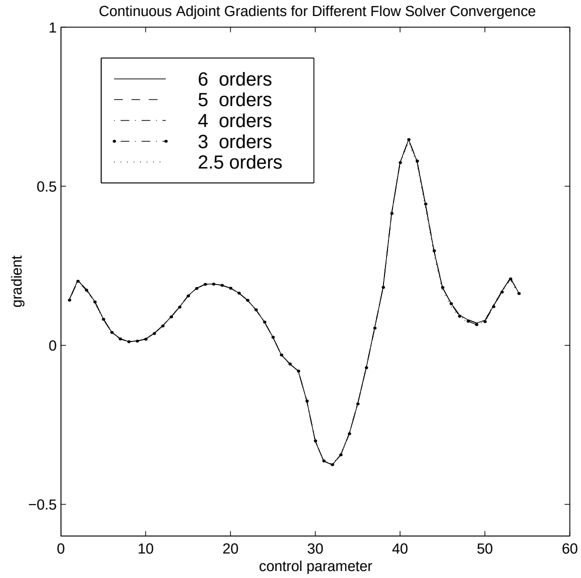
ADJOINT SOLVER CONVERGENCE HISTORY



Euler Inverse : Flow Convergence Issue

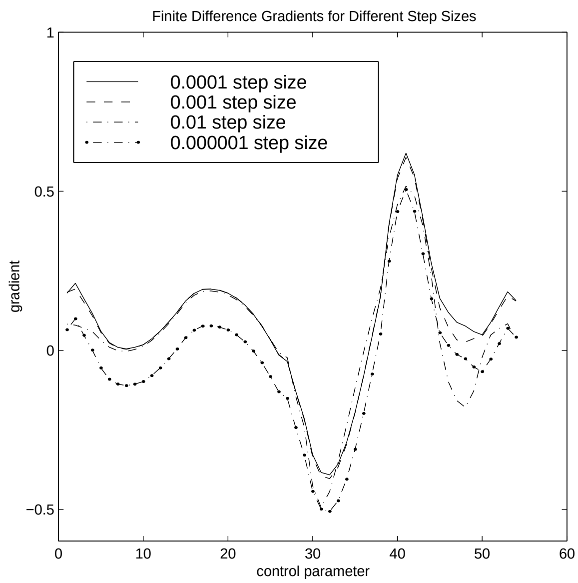


1a: Finite Difference

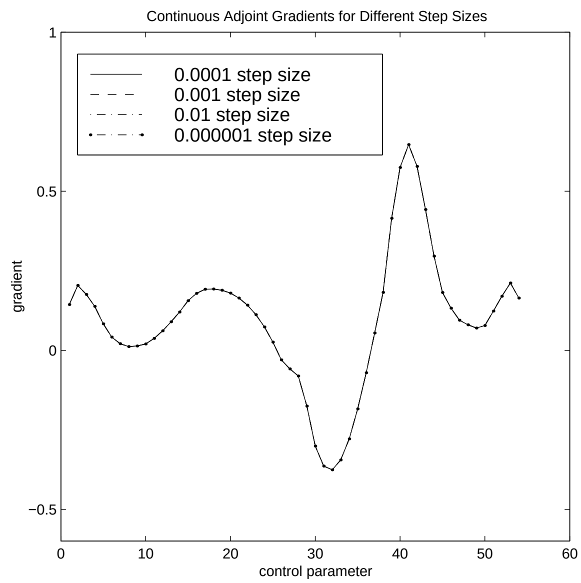


1b: Continuous Adjoint

Euler Inverse : Step Size Issue

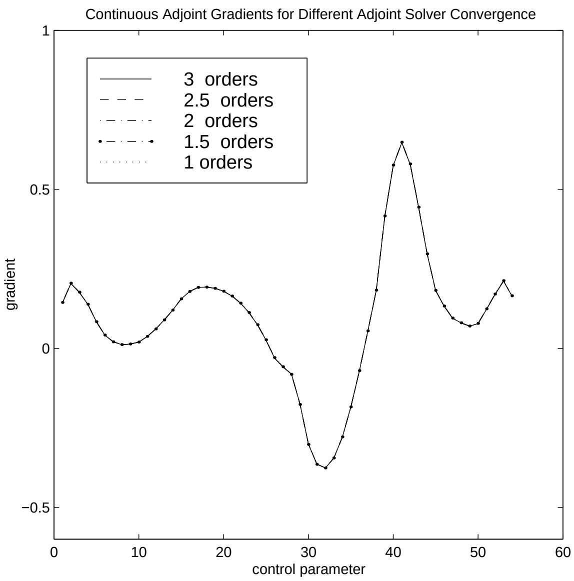


1c: Finite Difference

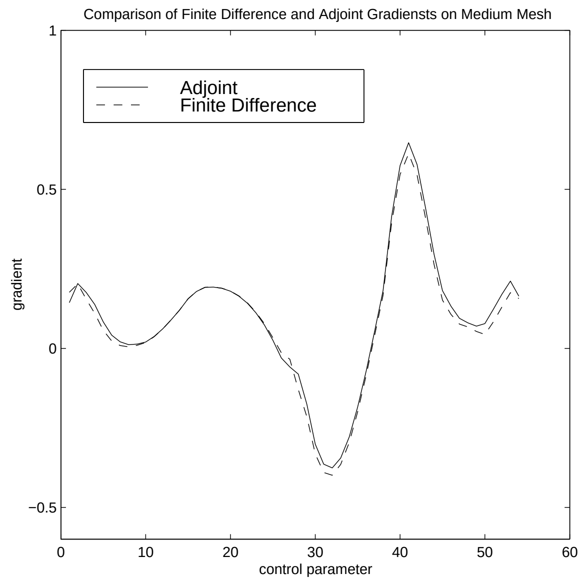


1d: Continuous Adjoint

Euler Inverse : Adjoint Convergence Issue
Finite Difference vs. Continuous Adjoint Gradients

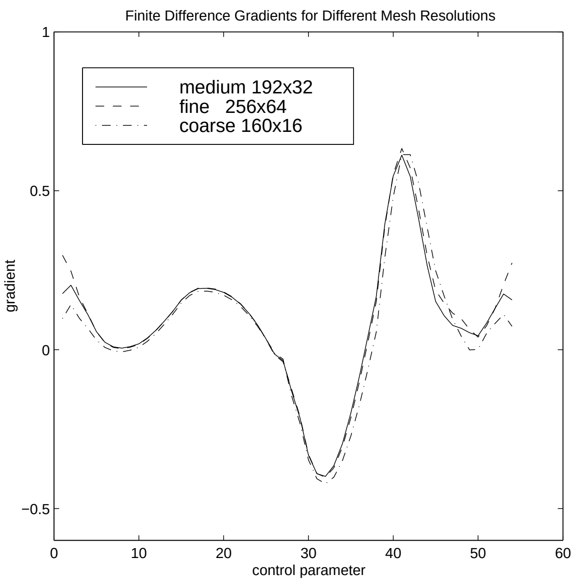


1e: Adjoint Converence Issue

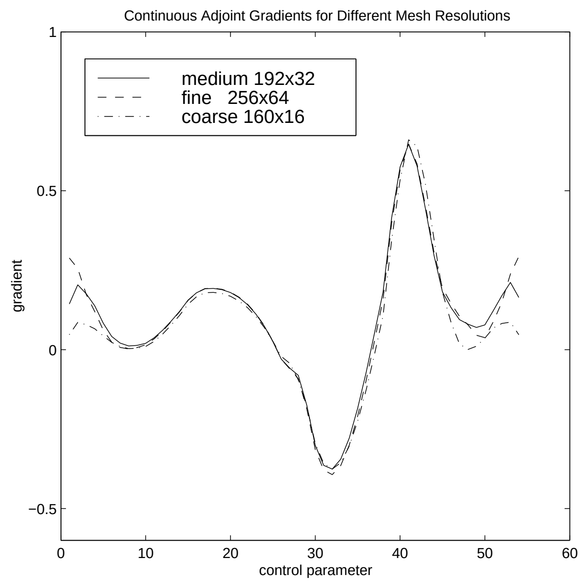


1f: Finite vs. Adjoint

Euler Inverse : Mesh Resolution Issue

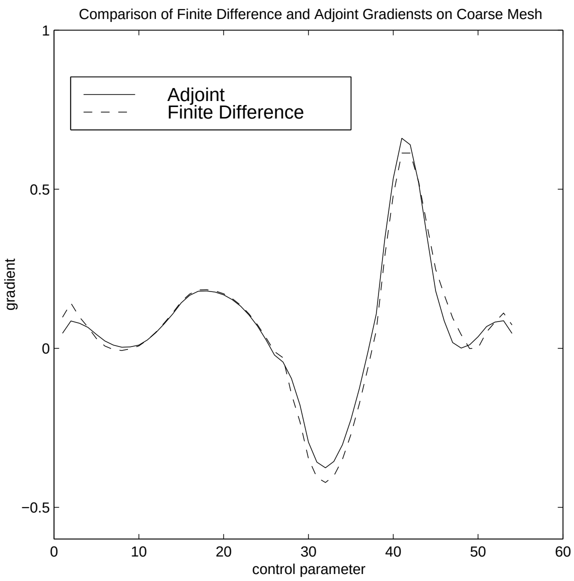


1g: Finite Difference

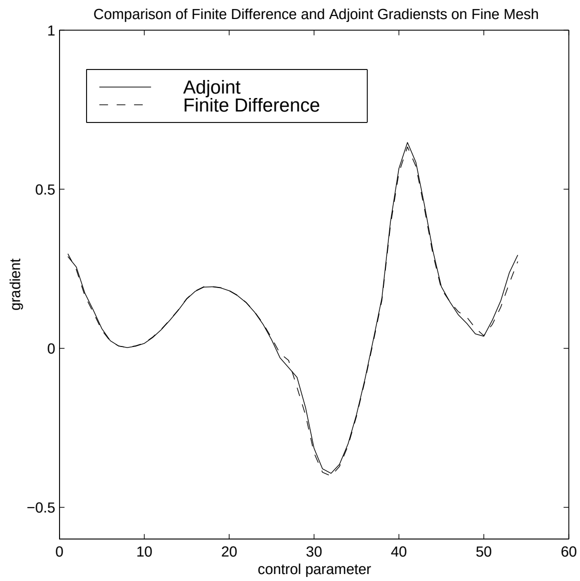


1h: Continuous Adjoint

Euler Inverse : Finite difference vs. Adjoint

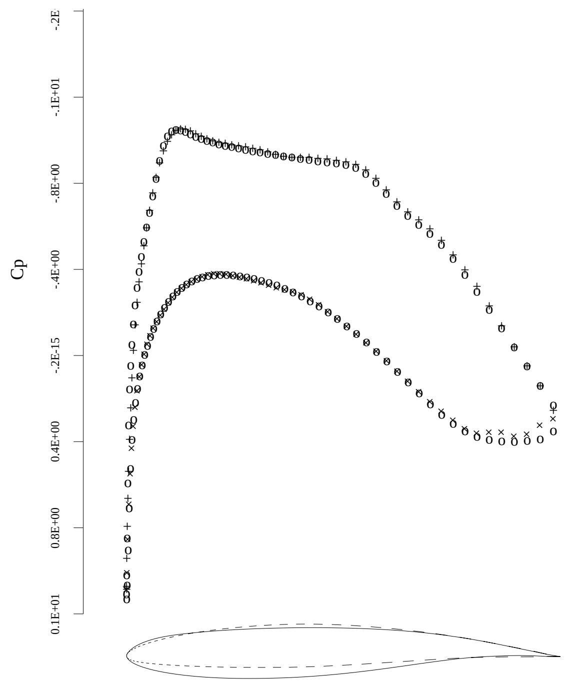


1i: On 160x16 Mesh



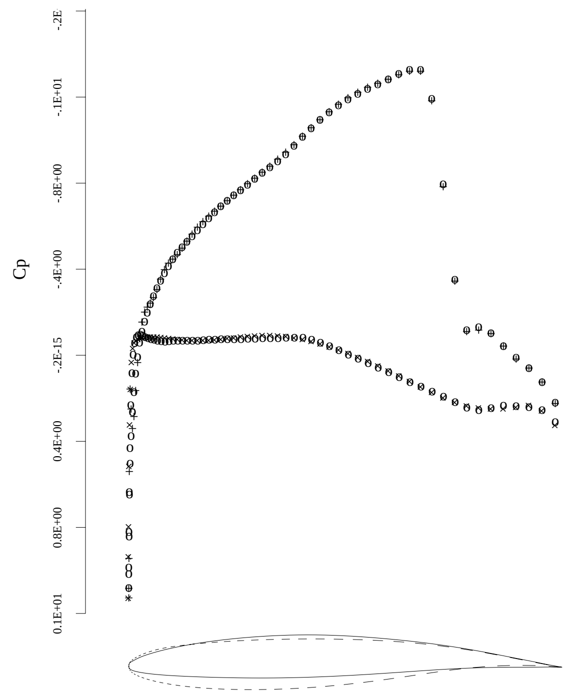
1j: On 256x64 Mesh

Euler Inverse : Design Examples



1k: NACA64A410 → Korn

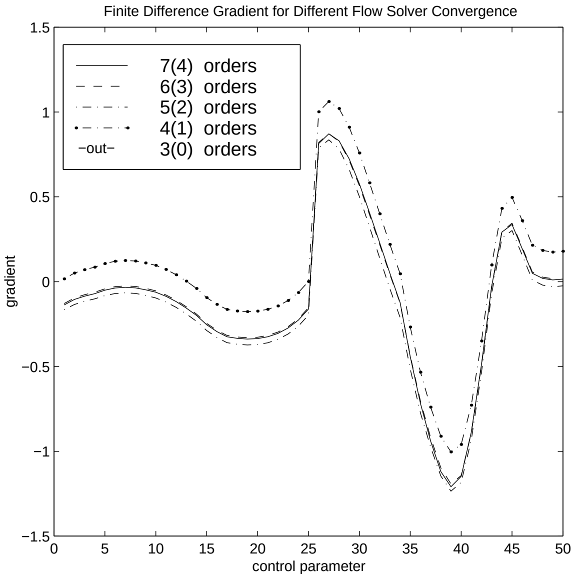
$M = 0.75, \alpha = 0.124$



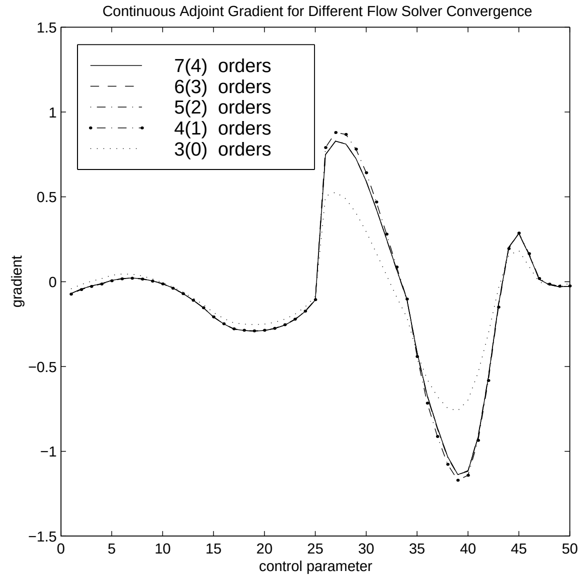
1l: Korn → NACA64A410

$M = 0.75, \alpha = 0.0$

Navier–Stokes Inverse : Flow Convergence Issue

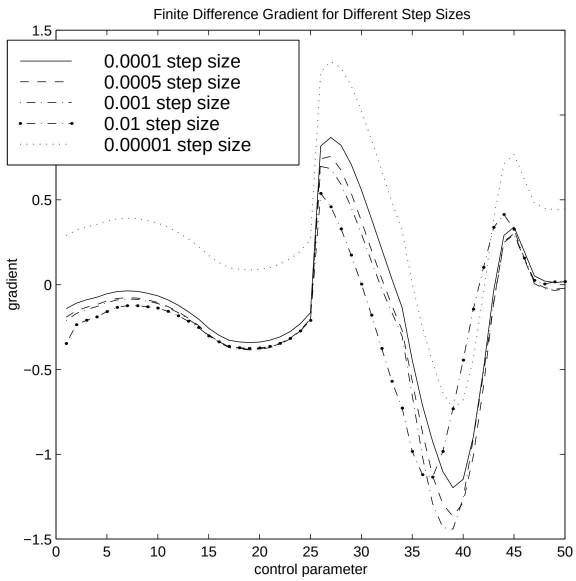


2a: Finite Difference

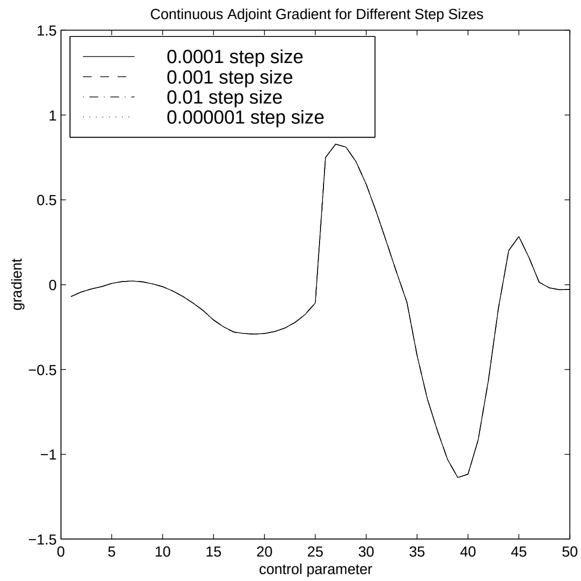


2b: Continuous Adjoint

Navier–Stokes Inverse : Step Size Issue

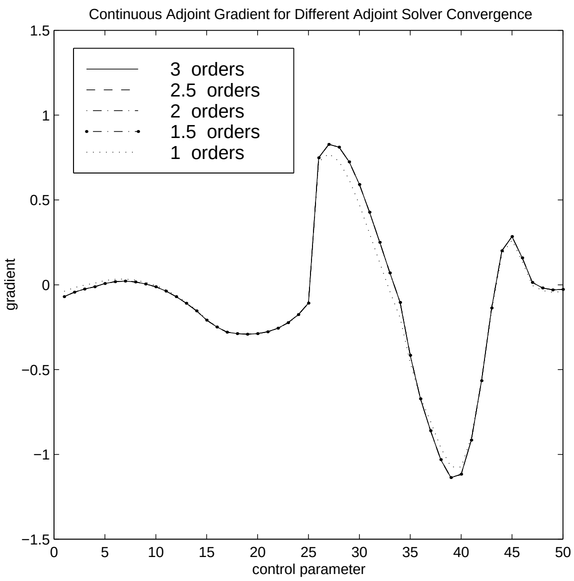


2c: Finite Difference

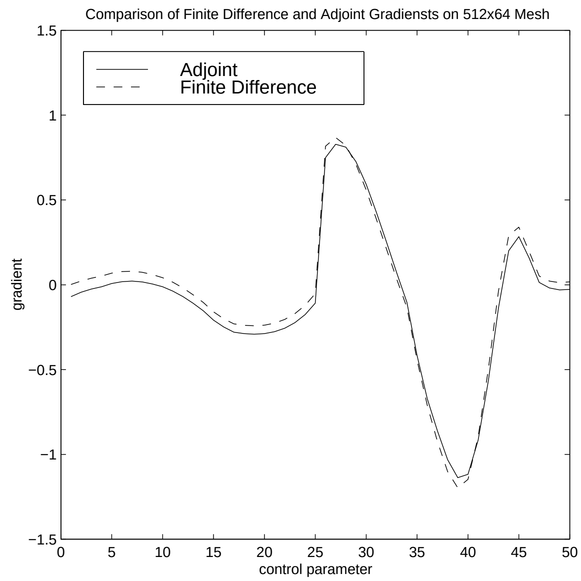


2d: Continuous Adjoint

Navier–Stokes Inverse : Adjoint Convergence Issue
Finite Difference vs. Continuous Adjoint Gradients

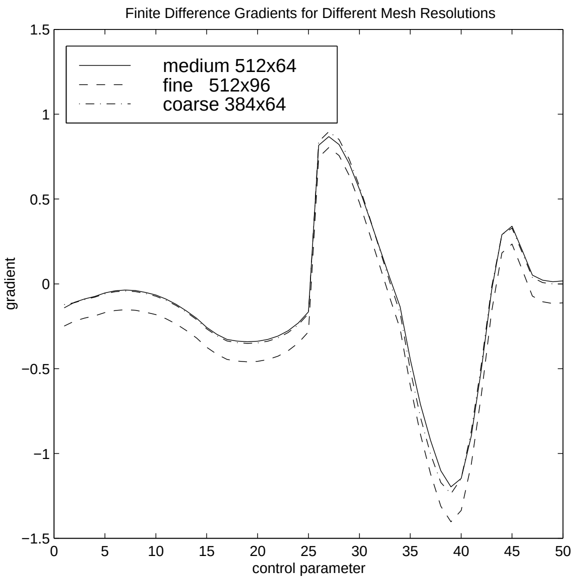


2e: Adjoint Convergence Issue

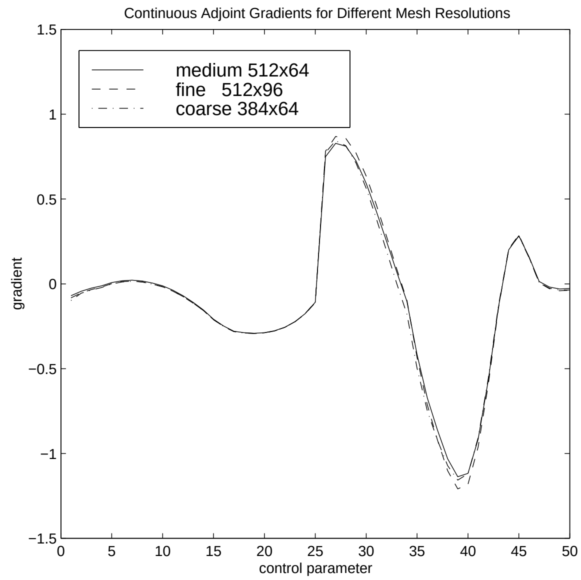


2f: Finite vs. Continuous Adjoint

Navier–Stokes Inverse : Mesh Resolution Issue

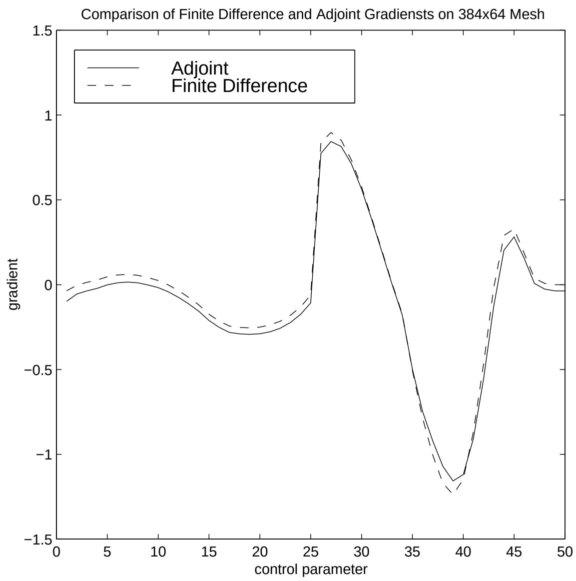


2g: Finite Difference

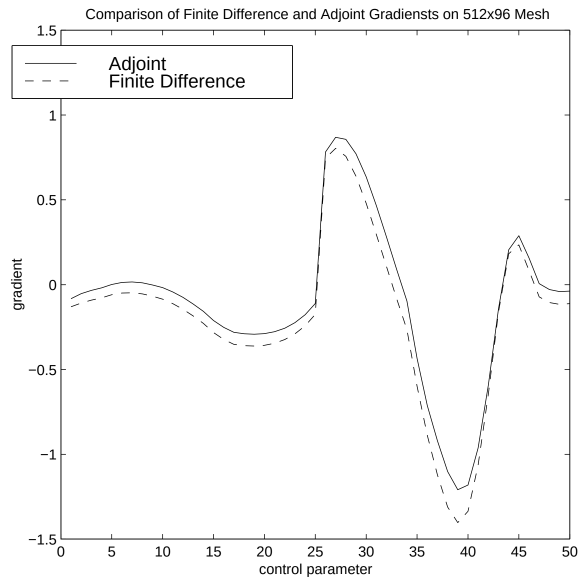


2h: Continuous Adjoint

Navier–Stokes Inverse : Finite Difference vs. Adjoint

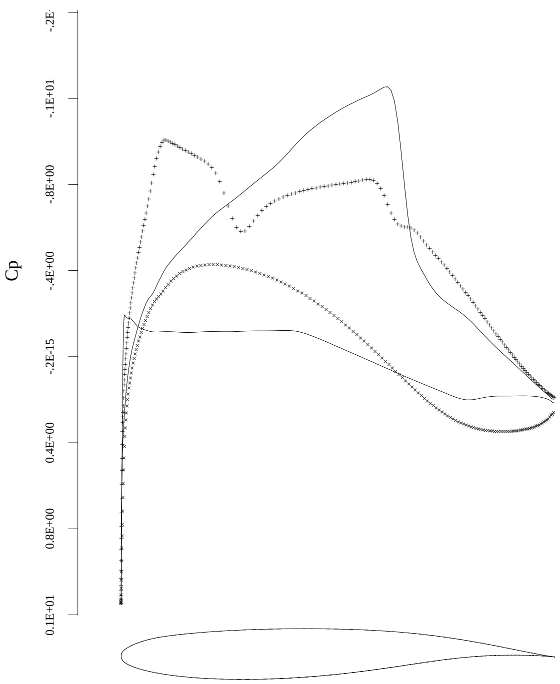


2i: On 384x64 Mesh

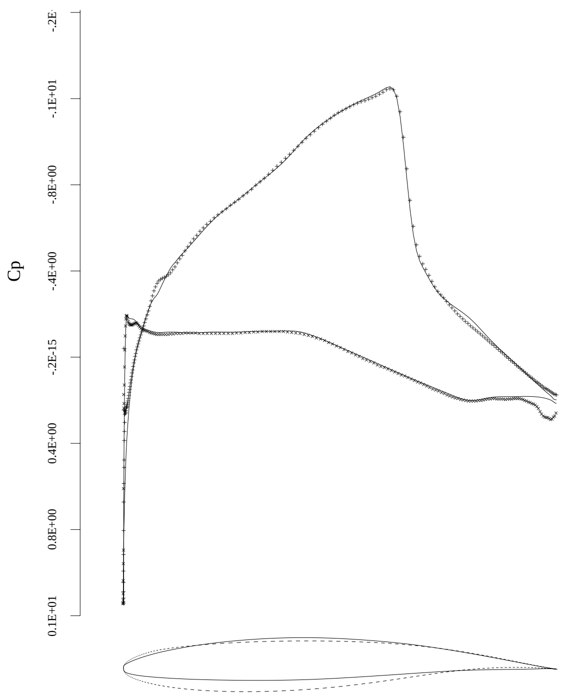


2j: On 512x96 Mesh

Navier–Stokes Inverse : KORN → NACA64A410

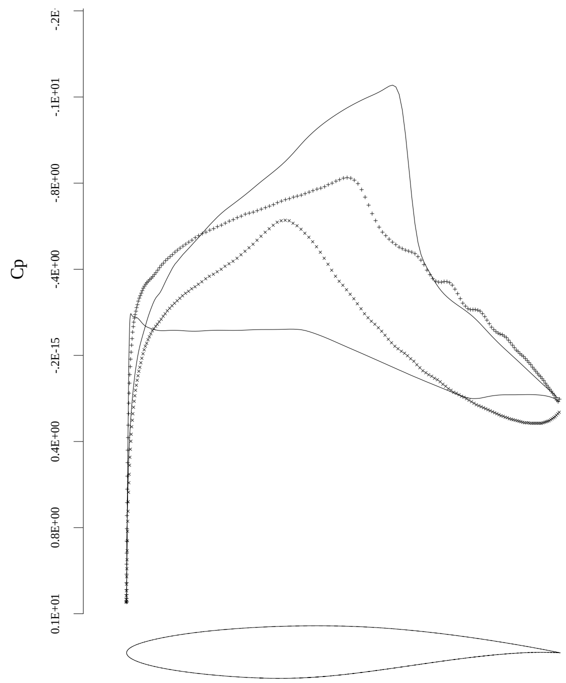


2k: Initial, $P_{error} = 0.0573$

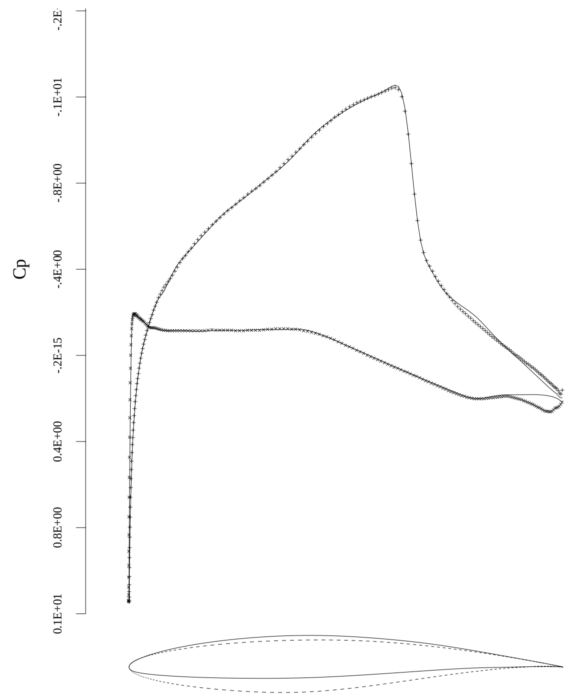


2l: 100 Design Iterations, $P_{error} = 0.0056$

Navier–Stokes Inverse : RAE2822 → NACA64A410

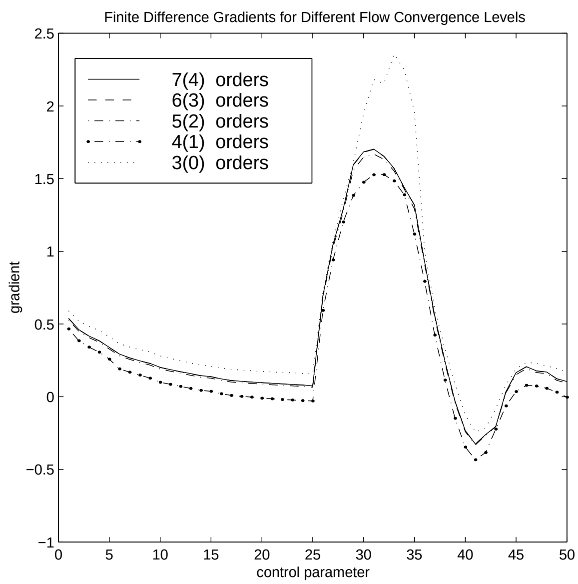


2m: Initial, $P_{error} = 0.0504$

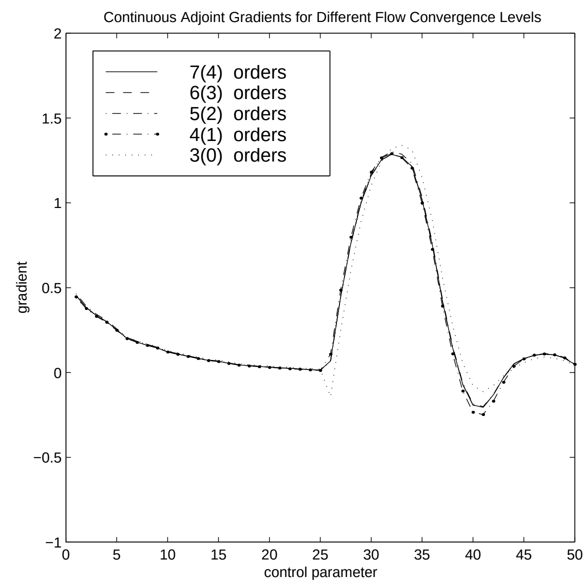


2n: 100 Design Iterations, $P_{error} = 0.0043$

Navier–Stokes CD_{total} Min. : Flow Convergence Issue

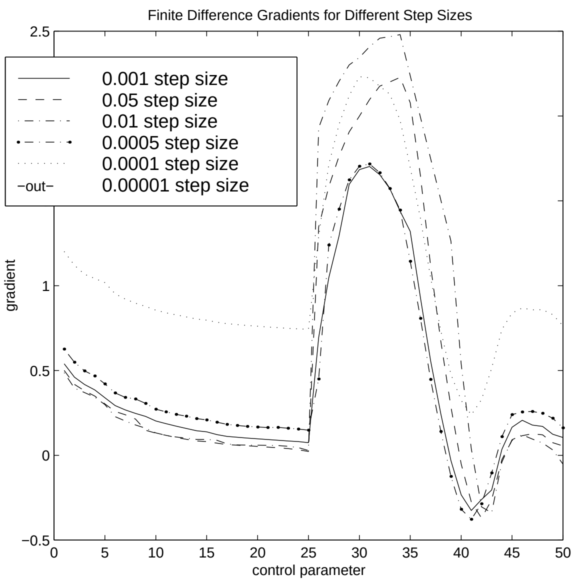


3a: Finite Difference

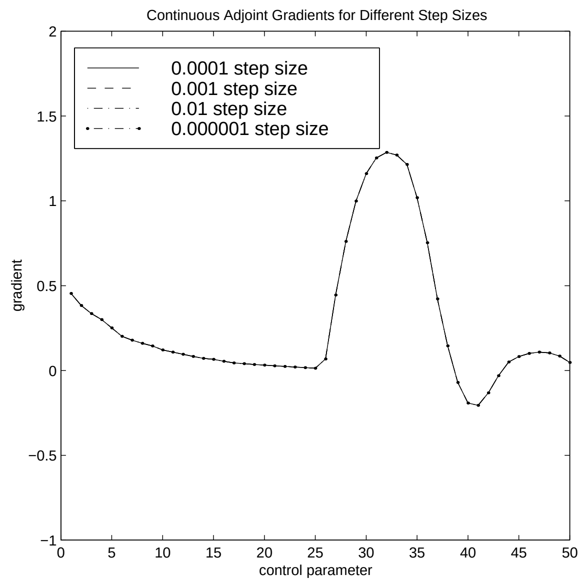


3b: Continuous Adjoint

Navier–Stokes CD_{total} Min. : Step Size Issue

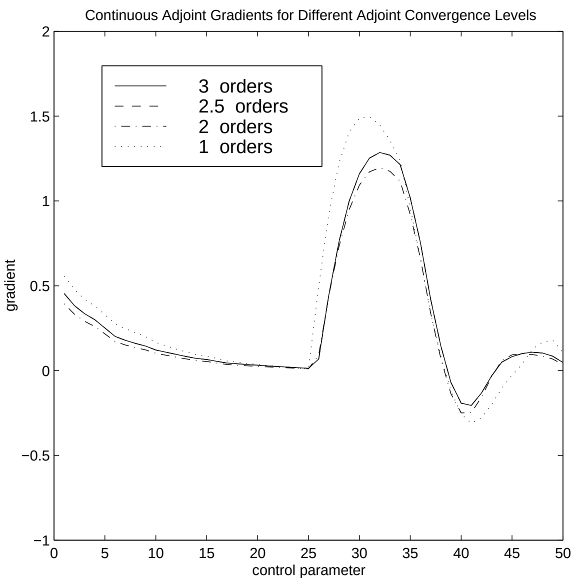


3c: Finite Difference

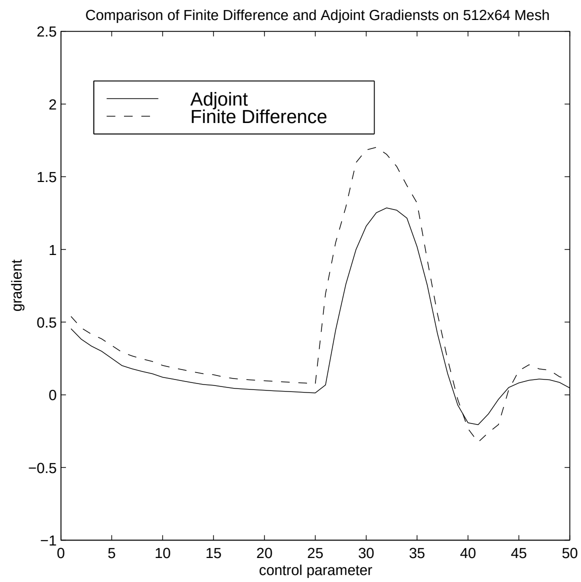


3d: Continuous Adjoint

Navier–Stokes CD_{total} Min. : Adjoit Solver Convergence Issue
Finite Difference vs. Continuous Adjoint Gradients

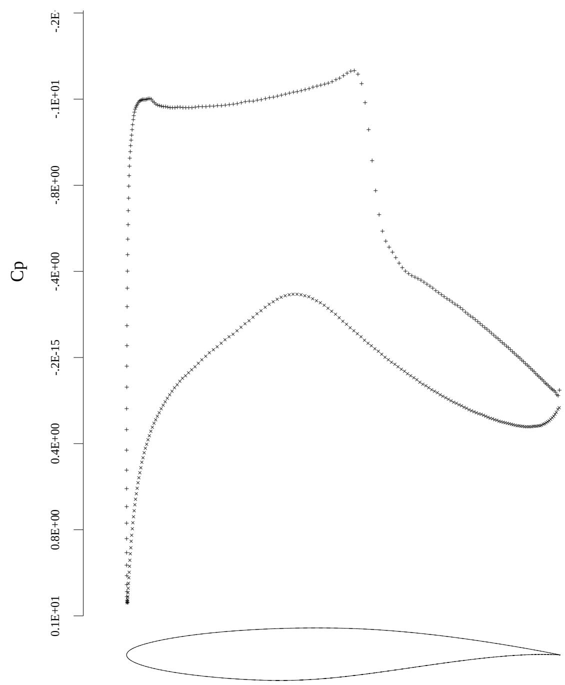


3e: Adjoint Convergence Issue

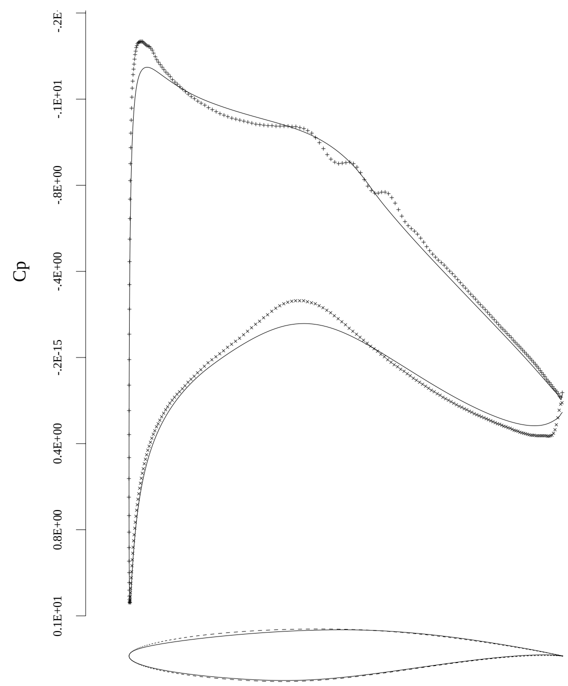


3f: Finite vs. Adjoint

Navier–Stokes CD_{total} Min. : RAE2822 at $\alpha = .84$

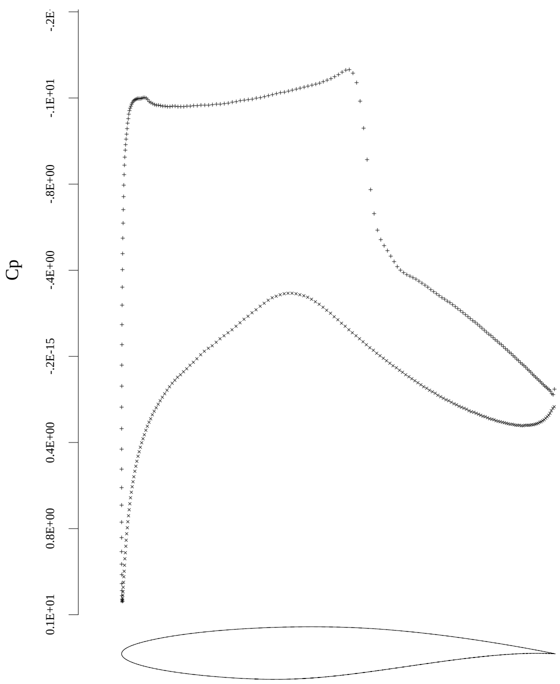


3g: Initial, CD_t=0.0168
$M = 0.73$, $\alpha = 2.756$, $CL_t = 0.8363$

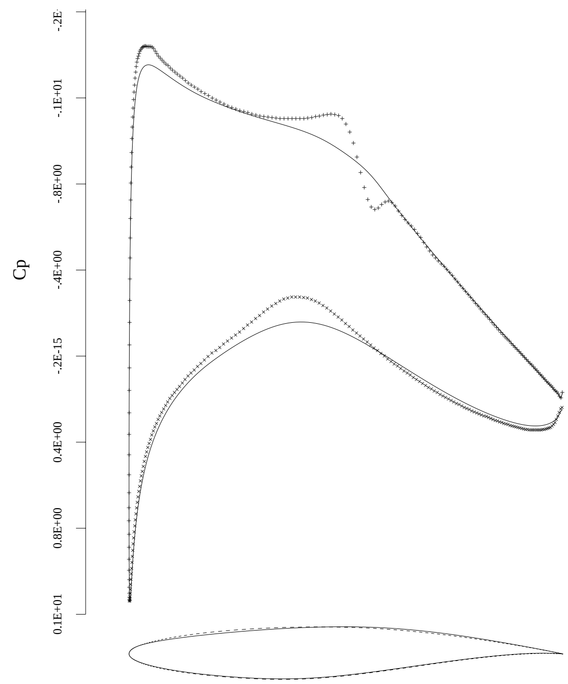


3h: 17 Design Iterations,CD_t=0.0096
$M = 0.73$, $\alpha = 2.565$, $CL_t = 0.8519$

Navier–Stokes $CD_{pressure}$ Min. : RAE2822 at $\alpha = .84$

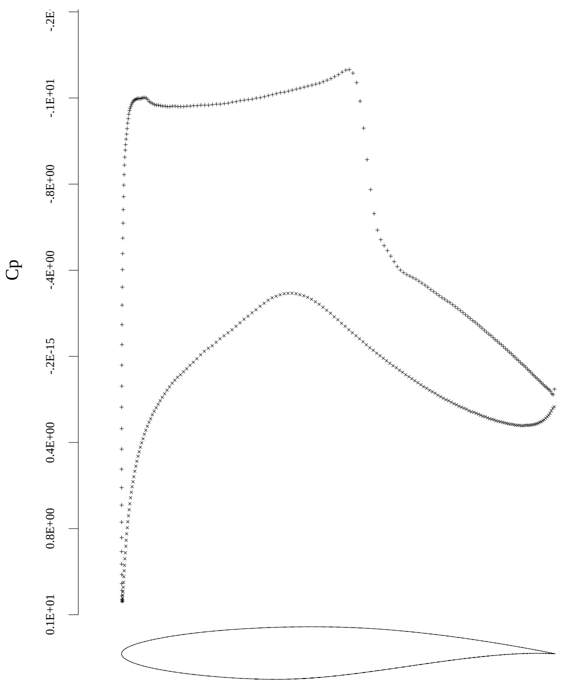


3i: Initial, CD_p=0.0115
$M = 0.73$, $\alpha = 2.756$, $CL_p = 0.8363$

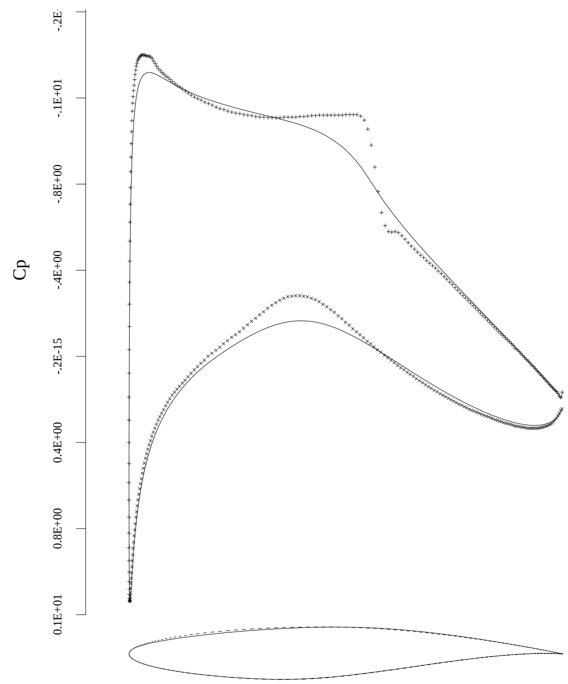


3j: 14 Design Iteration, CD_p=0.0042
$M = 0.73$, $\alpha = 2.557$, $CL_p = 0.8580$

Navier–Stokes $CD_{pressure}$ Min. : RAE2822 with CD_{total}

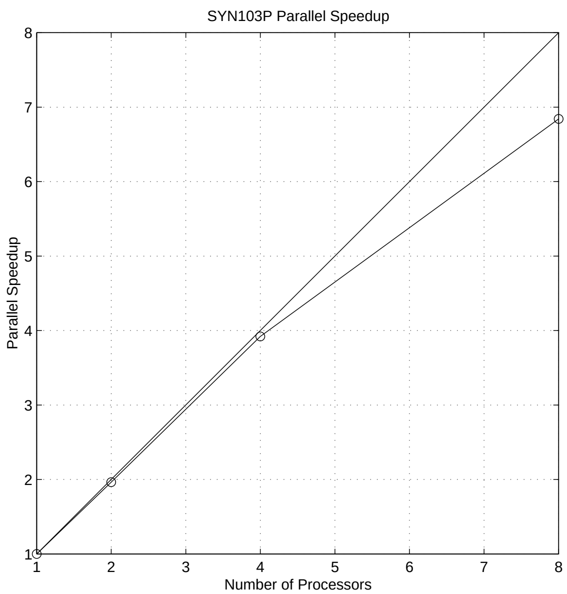


3k: Initial, CD_t=0.0168
$M = 0.73$, $\alpha = 2.756$, $CL_t = 0.8363$

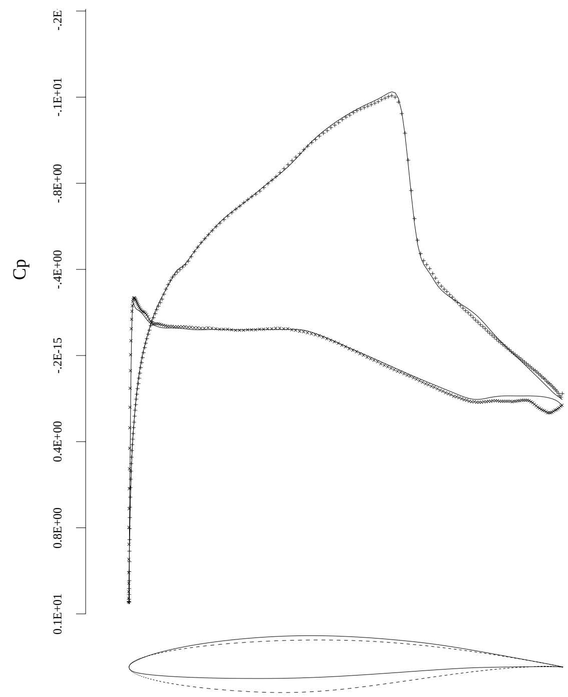


3l: 14 Design Iteration, CD_t=0.0108
$M = 0.73$, $\alpha = 2.665$, $CL_t = 0.8463$

MPI Speed Up & RAE2822 → NACA64A410



4a: Parallel Speedup



4b: RAE2822 → NACA64A410
100 Iterations, $P_{error} = 0.0068$