ARTIFICIAL DIFFUSION AND UPWINDING
FOR SYSTEMS OF CONSERVATION LAWS

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UPWIND BIASING FOR A SYSTEM AND FLUX DIFFERENCE SPLITTING

Suppose that a system of conservation laws
\[
\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} f(w) = 0
\]
is approximated by the semi-discrete scheme
\[
\Delta x \frac{dw_j}{dt} + h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}} = 0
\]
where the numerical flux is
\[
h_{j+\frac{1}{2}} = \frac{1}{2} (f_{j+1} + f_j) - d_{j+\frac{1}{2}}
\]
and \(d_{j+\frac{1}{2}}\) is a diffusive flux. Represent the mean value Jacobian matrix \(A(w) = \frac{\partial f}{\partial w}\) by \(A_{j+\frac{1}{2}}\) such that
\[
A_{j+\frac{1}{2}} (w_{j+1} - w_j) = f_{j+1} - f_j
\]
following the definition of Roe. Set
\[
A_{j+\frac{1}{2}} = T \Lambda T^{-1}
\]
where the eigenvectors of \(A\) are columns of \(T\) and \(\Lambda\) contains the eigenvalues. Now set
\[
d_{j+\frac{1}{2}} = \frac{1}{2} \left| A_{j+\frac{1}{2}} \right| (w_{j+1} - w_j)
\]
where
\[
\left| A_{j+\frac{1}{2}} \right| = T |\Lambda| T^{-1}
\]
GENERAL MODEL FOR NUMERICAL FLUX AND ARTIFICIAL DIFFUSION

The numerical flux is

\[ h_{j+\frac{1}{2}} = \frac{1}{2}(f_{j+1} + f_j) - d_{j+\frac{1}{2}} \]

Introduce Roe’s linearization

\[ f_{j+1} - f_j = A_{j+\frac{1}{2}}(w_{j+1} - w_j) \]

Define the diffusive flux as

\[ d_{j+\frac{1}{2}} = \frac{1}{2}B_{j+\frac{1}{2}}(w_{j+1} - w_j) \]

where \( B \) is a function of \( A \).

Expand \( B \) as a power series

\[ B = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \]

where higher powers are eliminated by the Cayley Hamilton theorem.
GENERAL MODEL FOR NUMERICAL FLUX AND ARTIFICIAL DIFFUSION
(continued)

Three main classes of schemes can thus be identified depending on the number of terms in the expansion

1 term: scalar diffusion

\[ d_{j+\frac{1}{2}} = \alpha_0 \Delta w_{j+\frac{1}{2}} \]

2 terms: CUSP, HLL, AUSM

\[ d_{j+\frac{1}{2}} = (\alpha_0 I + \alpha_1 A) \Delta w_{j+\frac{1}{2}} = \alpha_0 \Delta w_{j+\frac{1}{2}} + \alpha_1 \Delta f_{j+\frac{1}{2}} \]

3 terms: \( \alpha_0, \alpha_1, \) and \( \alpha_2 \) can be chosen so that \( B = |A| \) (characteristic splitting) by solving

\[ \alpha_0 + \alpha_1 \lambda_k + \alpha_2 \lambda_k^2 = |\lambda_k|, \quad k = 1, 2, 3 \]

Criteria for choosing coefficients for these schemes can be based on positivity and numerical shock structure.
## DISCRETE SHOCK STRUCTURE WITH CUSP SCHEMES

![Diagram](image)

The **numerical flux** is defined as

$$ h_{j+rac{1}{2}} = \frac{1}{2} (f_{j+1} + f_j) - d_{j+rac{1}{2}} $$

where the **diffusive flux** is

$$ d_{j+rac{1}{2}} = \frac{1}{2} \alpha^* c \Delta w_{j+rac{1}{2}} + \frac{1}{2} \beta \Delta f_{j+rac{1}{2}} $$

and in **supersonic** flow the scheme is **upwind**

$$ \alpha^* = 0 \quad \text{and} \quad \beta = \text{sign}(M) \quad \text{when} \ |M| \geq 1 $$

At the **entrance** to the **shock**

$$ h_{LB} = f_L \quad \text{because} \quad M_{LB} > 1 $$

maintaining equilibrium. At the **downstream side**

$$ h_{RR} = f_R \quad \text{(since} \ w \ \text{and} \ f \ \text{are constant)} $$

$$ h_{BR} = \frac{1}{2} (f_R + f_B) - \frac{1}{2} \beta (f_R - f_B) - \frac{1}{2} \alpha^* c (w_R - w_B) $$

Equating these

$$ (1 + \beta) (f_R - f_B) + \alpha^* c (w_R - w_B) = 0 $$

This is the **Hugoniot** condition for a shock speed

$$ \frac{-\alpha^* c}{1 + \beta} $$
DISCRETE SHOCK STRUCTURE WITH CUSP SCHEMES (continued)

Since

\[(f_R - f_B) = A_{BR} (w_R - w_B),\]

this is also an eigenvalue problem,

\[A_{BR}(w_R - w_B) + \frac{\alpha^* c}{1 + \beta} (w_R - w_B) = 0\]

For a solution \(\frac{-\alpha^* c}{1 + \beta}\) must be an eigenvalue of \(A_{BR}\).

When \(u > 0\) the only negative eigenvalue is \(u - c\).

Hence

\[\frac{\alpha^* c}{1 + \beta} = c - u\]

giving a 1 parameter family of solutions where \(\alpha^*\) determines \(\beta\).
**E-CUSP SCHEME**
(Int J. of CFD, 5, 1998, 1-38)

The **diffusive flux** formed as a combination of differences of the **state and flux vectors**

\[ d_{j+\frac{1}{2}} = \frac{1}{2} \alpha_{j+\frac{1}{2}}^* (w_{j+1} - w_j) + \frac{1}{2} \beta_{j+\frac{1}{2}} (f_{j+1} - f_j) \]

**Full upwinding in supersonic flow** is obtained by setting

\[ \alpha_{j+\frac{1}{2}}^* = 0, \quad \beta_{j+\frac{1}{2}} = \text{sign}(M) \] when \(|M| > 1\)

To support a **stationary discrete shock structure** with a **single interior point** \(\alpha^*\) and \(\beta\) cannot be chosen independently.

To satisfy equilibrium at the shock exit

\[ \alpha^* = (1 + \beta) (1 - M) \] when \(M > 0\)

The choice \(\beta = M\) when \(|M| < 1\) corresponds to the **HLL scheme**, which is very diffusive.
E-CUSP SCHEME (continued)

To produce a less diffusive scheme split the flux vectors as

\[ f = uw + f_p, \quad f_p = \begin{pmatrix} 0 \\ p \\ up \end{pmatrix} \]

Then

\[ f_{j+1} - f_j = \bar{u} (w_{j+1} - w_j) + \bar{w} (u_{j+1} - u_j) + f_{p_{j+1}} - f_{p_j} \]

where \( \bar{u} \) and \( \bar{w} \) are arithmetic averages.

The effective coefficient of convective diffusion is thus

\[ \alpha c = \alpha^* c + \beta \bar{u} \]

Take

\[ \alpha = |M| \]

Then

\[ \beta = \begin{cases} \text{sign}(M) & , \quad |M| \geq 1 \\ \max(2M - 1, 0) & , \quad 0 \leq M \leq 1 \\ \min(2M + 1, 0) & , \quad -1 \leq M \leq 0 \end{cases} \]
CHARACTERISTIC DECOMPOSITION OF THE E-CUSP SCHEME

The diffusive flux is

\[ d_{j+\frac{1}{2}} = \frac{1}{2} \alpha^* c \Delta w_{j+\frac{1}{2}} + \frac{1}{2} \beta \Delta f_{j+\frac{1}{2}} \]

\[ = \frac{1}{2} (\alpha^* c I + \beta A_{j+\frac{1}{2}}) \Delta w_{j+\frac{1}{2}} \]

where

\[ A_{j+\frac{1}{2}} \Delta w_{j+\frac{1}{2}} = \Delta f_{j+\frac{1}{2}} \]

with the characteristic decomposition

\[ A_{j+\frac{1}{2}} = R \Lambda R^{-1}, \quad \Lambda = \text{diag} (u, u + c, u - c) \]

Thus

\[ d_{j+\frac{1}{2}} = RM R^{-1}, \quad M = \text{diag} (\mu_1 c, \mu_2 c, \mu_3 c) \]

where

\[ \mu_1 = \alpha^* + \beta M \]
\[ \mu_2 = \alpha^* + \beta (M + 1) \]
\[ \mu_3 = \alpha^* + \beta (M - 1) \]

Also

\[ \alpha^* = \alpha - \beta M = |M| - \beta M \]
The scheme is less diffusive than the characteristic upwind scheme.
HIGHER ORDER CUSP SCHEME

Define left and right states at the interface $j + \frac{1}{2}$

$$w_L = w_j + \frac{1}{2}L \left( \Delta w_{j+\frac{3}{2}}, \Delta w_{j-\frac{1}{2}} \right)$$

$$w_R = w_{j+1} - \frac{1}{2}L \left( \Delta w_{j+\frac{3}{2}}, \Delta w_{j-\frac{1}{2}} \right)$$

with $f_L = f \left( w_L \right)$

$$f_R = f \left( w_R \right)$$

Then the numerical flux is

$$h_{j+\frac{1}{2}} = \frac{1}{2} (f_R + f_L) - \frac{1}{2} \alpha^* c (w_R - w_L) - \frac{1}{2} \beta (f_R - f_L)$$

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Artificial Diffusion and Upwinding for Systems of Conservation Laws