Inverse Filtering for Compressible Flows

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The instantaneous equations are:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0
\]  \hspace{1cm} (1)

\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial x_j}
\]  \hspace{1cm} (2)

\[
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u_j H)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\rho u_i \sigma_{ij} + k \frac{\partial T}{\partial x_j}\right)
\]  \hspace{1cm} (3)

Let \( P \) be an invertible filter with inverse \( Q \) such that,

Commute with derivatives,

\[
P \frac{\partial}{\partial x_i} = \frac{\partial P}{\partial x_i}, \quad Q \frac{\partial}{\partial x_i} = \frac{\partial Q}{\partial x_i}
\]  \hspace{1cm} (4)

Note that \( P \) is not a projector such that,

\[
P^2 = P
\]

If this were so, then

\[
P^2 Q = PQ = I
\]

and

\[
P^2 Q = P(PQ) = P
\]

so,

\[
P = I
\]

Define filtered quantities,

\[
\bar{\rho} = P\rho, \rho = Q\bar{\rho}, \bar{p} = Pp, p = Q\bar{p}
\]  \hspace{1cm} (5)
and mass weighted filtered velocity

$$\bar{u}_i = \frac{P(\rho u_i)}{\rho} = \frac{P(\rho u_i)}{\bar{\rho}} \quad (6)$$

$$\bar{\rho} \bar{u}_i = P(\rho u_i), \rho u_i = Q(\bar{\rho} \bar{u}_i) \quad (7)$$

Then substituting for $\rho$ and $\rho u_i$, the mass conservation equation becomes,

$$\frac{\partial}{\partial t} Q \bar{\rho} + \frac{\partial}{\partial x_j} Q(\bar{\rho} \bar{u}_i) = 0 \quad (8)$$

and multiplying by $P = Q^{-1}$,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_i) = 0 \quad (9)$$

The filtered Momentum equations are,

$$\frac{\partial}{\partial t} P(\rho u_i) + \frac{\partial}{\partial x_j} P(\rho u_i u_j) + \frac{\partial}{\partial x_i} P \rho = \frac{\partial}{\partial x_j} P(\sigma_{ij}) \quad (10)$$

substituting for $\rho u_i$ and $p$,

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{u}_i) + \frac{\partial}{\partial x_j} P(\rho u_i u_j) + \frac{\partial}{\partial x_i} \bar{\rho} = \frac{\partial}{\partial x_j} P(\bar{\sigma}_{ij}) \quad (11)$$

where $\bar{\sigma}_{ij}$ is evaluated from $\bar{u} = Pu_i$.

To reduce this to the usual form,

$$\rho u_i u_j = \frac{Q(\bar{\rho} \bar{u}_i)Q(\bar{\rho} \bar{u}_j)}{Q\bar{\rho}} \quad (12)$$

so,

$$\frac{\partial}{\partial x_j} P(\rho u_i u_j) = \frac{\partial}{\partial x_j} P(\bar{\rho} \bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j} \bar{\tau}_{ij} \quad (13)$$

where,

$$\bar{\tau}_{ij} = Q^{-1} \left[ \frac{Q(\bar{\rho} \bar{u}_i)Q(\bar{\rho} \bar{u}_j)}{Q\bar{\rho}} - Q(\bar{\rho} \bar{u}_i \bar{u}_j) \right] \quad (14)$$

Also,

$$\frac{\partial}{\partial x_j} \bar{\sigma}_{ij} = \frac{\partial}{\partial x_j} (\bar{\sigma}_{ij} + \eta_{ij}) \quad (15)$$

where,

$$\bar{\sigma}_{ij} = \mu \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial \bar{u}_k}{\partial x_k} \quad (16)$$
and \( \eta_{ij} \) is the same quantity evaluated with,

\[
\bar{u}_i - \bar{u}_i = \frac{\bar{m}_i}{\bar{\rho}} - \bar{u}_i
\]

(17)

where,

\[
m_i = \rho u_i, \quad \bar{m}_i = P(\rho u_i)
\]

then,

\[
\bar{u}_i = P \frac{m_i}{\rho} = Q^{-1} \left( \frac{Q\bar{m}_i}{\bar{\rho}} \right)
\]

(18)

so,

\[
\bar{u}_i - \bar{u}_i = Q^{-1} \left[ \frac{\bar{m}_i}{\bar{\rho}} - \frac{Q\bar{m}_i}{\bar{\rho}} \right]
\]

(20)

Finally, the momentum equation is,

\[
\frac{\partial}{\partial t} (\bar{\rho} \bar{u}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_i \bar{u}_j) + \frac{\partial \bar{p}}{\partial x_i} = \frac{\partial}{\partial x_j} (\tau_{ij} + \eta_{ij})
\]

(21)

Here,

\[
\tau_{ij} = Q^{-1} \left[ \frac{Q(\bar{\rho} \bar{u}_i)Q(\bar{\rho} \bar{u}_j)}{\bar{\rho}} - Q(\bar{\rho} \bar{u}_i \bar{u}_j) \right]
\]

(22)

\[
\tau_{ij} = Q^{-1} \left[ \frac{Q\bar{m}_iQ\bar{m}_j}{\bar{\rho}} - Q\left( \bar{m}_i \bar{m}_j \right) \right]
\]

(23)

The filtered energy equations are,

\[
\frac{\partial}{\partial t} P(\rho E) + \frac{\partial}{\partial x_j} \left[ P(\rho u_j H) \right] = \frac{\partial}{\partial x_j} \left[ P(u_i \sigma_{ij} + k \frac{\partial T}{\partial x_j}) \right]
\]

(24)

Define,

\[
\bar{\rho} \bar{E} = P(\rho E) \quad \frac{P(\rho E)}{\rho} = \bar{\rho} \bar{E}
\]

(25)

\[
\rho \bar{E} = Q(\bar{\rho} \bar{E})
\]

(26)

\[
\bar{\rho} \bar{H} = \bar{\rho} \bar{E} + \bar{\rho}, \quad Q(\bar{\rho} \bar{H}) = \rho E + p
\]

(27)

Then,

\[
\rho u_j H = \frac{\rho u_j \rho H}{\rho} = \frac{Q(\bar{\rho} \bar{u}_i)Q(\bar{\rho} \bar{H})}{\bar{\rho}}
\]

(28)

\[
u_i \sigma_{ij} = \frac{\rho u_i \sigma_{ij}}{\rho} = \frac{Q(\bar{\rho} \bar{u}_i)Q \sigma_{ij}}{\bar{\rho}} = \frac{Q(\bar{\rho} \bar{u}_i)Q(\sigma_{ij} + \eta_{ij})}{\bar{\rho}}
\]

(29)

\[
P(\rho u_j H) = Q^{-1} \left[ \frac{Q(\bar{\rho} \bar{u}_j)Q(\bar{\rho} \bar{H})}{\bar{\rho}} \right]
\]

(30)
\[ P(u, \sigma_{ij}) = Q^{-1} \left[ \frac{Q(\tilde{\rho} \tilde{u}_i)Q(\sigma_{ij} + \eta_{ij})}{\tilde{\rho}} \right] \] (31)

Thus, the energy equations can be written as,

\[
\frac{\partial}{\partial t}(\tilde{\rho} \tilde{E}) + \frac{\partial}{\partial x_j}(\tilde{\rho} \tilde{u}_j \tilde{H}) = \frac{\partial}{\partial x_j}(\tilde{u}_i \sigma_{ij} + k \frac{\partial \tilde{T}}{\partial x_j}) - \frac{\partial}{\partial x_j}(\alpha_{ij} + \beta_{ij}) \] (32)

Where,

\[
\alpha_{ij} = Q^{-1} \left[ \frac{Q(\tilde{\rho} \tilde{u}_i)Q(\tilde{\rho} \tilde{H})}{\tilde{\rho}} - Q(\tilde{\rho} \tilde{u}_j \tilde{H}) \right] \] (33)

and

\[
\beta_{ij} = Q^{-1} \left[ \frac{Q(\tilde{\rho} \tilde{u}_i)Q(\sigma_{ij} + \eta_{ij})}{\tilde{\rho}} - Q(\tilde{\rho} \tilde{u}_i \sigma_{ij}) \right] \] (34)